Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 9**

Homological Algebra

Problem 1: Suppose that $\pi : F \to G$ is a surjective group homomorphism and that $\eta : \mathbb{Z}F \to \mathbb{Z}G$ is its linear extension to a ring homomorphism between the group rings. Let $N := \ker(\pi)$ be the kernel of π and I_N be its augmentation ideal, i.e., the kernel of the augmentation $\varepsilon_N : \mathbb{Z}N \to \mathbb{Z}$. Analogously, let I_F be the augmentation ideal of F, i.e., the kernel of the augmentation $\varepsilon_F : \mathbb{Z}F \to \mathbb{Z}$.

Suppose furthermore that G is finite, that F is a free group on the finite set $X \subset F$, and that N is a free group on the finite set $Y \subset N$.

We define $A_0 := I_F$, $B_0 := (\mathbb{Z}F)I_N$, and for $m = 1, 2, 3, \ldots$ recursively $B_m := B_{m-1}B_0$ as well as $A_m := I_F B_{m-1}$.

1. Show that B_m is a free right $\mathbb{Z}F$-module. (5 points)
2. Show that A_m is a free right $\mathbb{Z}F$-module. (5 points)
3. Show that B_m/B_{m+1} is a free right $\mathbb{Z}G$-module. (5 points)
4. Show that A_m/A_{m+1} is a free right $\mathbb{Z}G$-module. (5 points)
5. Show that

$$\ldots \longrightarrow B_1/B_2 \longrightarrow A_1/A_2 \longrightarrow B_0/B_1 \longrightarrow A_0/A_1 \longrightarrow \mathbb{Z}G \xrightarrow{\varepsilon_G} \mathbb{Z}$$

is a free resolution of the trivial right $\mathbb{Z}G$-module $\mathbb{Z}$. (5 points)

(Hint: Use the results from Sheet 8. As we said there, it follows from the Nielsen-Schreier theorem that N is a free group, so that this hypothesis is not necessary.)

Problem 2: Suppose that R is a ring, that M is a right R-module, that N is a left R-module, and that

$$\cdots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} M$$

is a projective resolution of M. The homology groups of the chain complex

$$\cdots \xrightarrow{d_3 \otimes_R \mathrm{id}_N} X_2 \otimes_R N \xrightarrow{d_2 \otimes_R \mathrm{id}_N} X_1 \otimes_R N \xrightarrow{d_1 \otimes_R \mathrm{id}_N} X_0 \otimes_R N \longrightarrow \{0\} \longrightarrow \ldots$$

are denoted by $\mathrm{Tor}_m^R(M, N)$. Show that $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$. (25 points)

(Hint: Use Problem 4 on Sheet 5. Although the notation does not reflect it, the above definition depends on the resolution. It follows from the comparison theorem that different resolutions lead to isomorphic homology groups.)

Problem 3: Suppose that R is a ring, that M is a right R-module, and that $N \xrightarrow{f} P \xrightarrow{g} Q$ is a short exact sequence of left R-modules. Show that there is a long exact sequence

$$\cdots \xrightarrow{\partial_{m+1}} \operatorname{Tor}_m^R(M,N) \xrightarrow{f_m} \operatorname{Tor}_m^R(M,P) \xrightarrow{g_m} \operatorname{Tor}_m^R(M,Q) \xrightarrow{\partial_m} \operatorname{Tor}_{m-1}^R(M,N) \longrightarrow \cdots$$

where f_m is the map that the chain map $(\mathrm{id}_{X_m} \otimes_R f)$ induces in homology. The map g_m is defined similarly, and the map ∂_m is a connecting homomorphism. (25 points)

(Hint: Use Problem 4 on Sheet 7 to construct a short exact sequence of chain complexes, and apply the long exact homology sequence.)

Problem 4: In the group $\mathrm{SL}(2,\mathbb{Z}_3)$, consider the matrices

$$\sigma := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \tau := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \rho := \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

It is easy to verify that these matrices satisfy the following relations, which you can use without proof:

$$\sigma^4 = \rho^3 = 1 \qquad \rho^{-1}\sigma\rho = \tau \qquad \rho^{-1}\tau\rho = \sigma\tau \qquad \sigma = \rho^{-1}(\sigma\tau)\rho = \tau\sigma\tau \qquad \tau^{-1}\sigma\tau = \sigma^{-1}$$

Writing $\mathbb{Z}_4 = \{0,1,2,3\}$, we set $H := \mathrm{SL}(2,\mathbb{Z}_3) \times \mathbb{Z}_4$ and $N := \langle(\sigma^2, 2)\rangle \subset H$. Furthermore, we set $G := H/N$ and let Q be the subgroup of G generated by the equivalence classes of $(\sigma, 0)$ and $(\tau, 0)$.

1. Show that σ^2 is central in $\mathrm{SL}(2,\mathbb{Z}_3)$. (2 points)
2. Show that $|N| = 2$ and that N is a normal subgroup of H. (2 points)
3. Show that $|G| = 48 = 2^4 \cdot 3$. (5 points)
4. Show that $|Q| = 8$ and that Q is a normal subgroup of G. (5 points)
5. Find a 2-Sylow subgroup S_2 of G that contains Q. (5 points)
6. Find a complement of Q in S_2. (3 points)
7. Find a 3-Sylow subgroup S_3 of G. (2 points)
8. Show that, for any Sylow subgroup S of G, the subgroup $S \cap Q$ has a complement in S. (1 point)

(Remark: It can be shown that Q does not have a complement in G. So the existence of a complement cannot be inferred from the existence of a complement within the Sylow subgroups, in contrast to the case where the normal subgroup is abelian, which we discussed in class.)

Due date: Wednesday, April 7, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.