Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 8**

Homological Algebra

Problem 1: Suppose that F is a free group on the finite set $X \subset F$ of free generators, and that $\mathbb{Z}F$ is its group ring. We have seen in class that, for each $x \in X$, there is a unique derivation $D_x : F \to \mathbb{Z}F$ satisfying $D_x(x) = 1$ and $D_x(y) = 0$ for $y \in X$ with $y \neq x$. Show that we have

$$w - 1 = \sum_{x \in X} (x - 1) D_x(w)$$

for all $w \in F$. (25 points)

(Hint: Recall that the left-hand side of this equation is the universal derivation that we discussed in class.)

Problem 2: Suppose that $\pi : F \to G$ is a surjective group homomorphism and that $\eta : \mathbb{Z}F \to \mathbb{Z}G$ is its linear extension to a ring homomorphism between the group rings. Let $N := \ker(\pi)$ be the kernel of π and I_N be its augmentation ideal, i.e., the kernel of the augmentation $\varepsilon_N : \mathbb{Z}N \to \mathbb{Z}$.

1. Show that
$$\ker(\eta) = (\mathbb{Z}F)I_N = I_N(\mathbb{Z}F)$$
where $(\mathbb{Z}F)I_N = \mathrm{Span}_{\mathbb{Z}}(\{ab \mid a \in \mathbb{Z}F, b \in I_N\})$, and $I_N(\mathbb{Z}F)$ is defined similarly. (10 points)

2. Suppose that G is finite, that F is a free group on the finite set $X \subset F$, and that N is a free group on the finite set $Y \subset N$. Show that $(\mathbb{Z}F)I_N$ is a free left (or right) $\mathbb{Z}F$-module with the basis $\{y - 1 \mid y \in Y\}$. (15 points)

(Remark: Recall that we have proved in class that I_N is a free right $\mathbb{Z}N$-module with basis $\{y - 1 \mid y \in Y\}$. A similar argument, which you do not have to carry out, shows that I_N is a free left $\mathbb{Z}N$-module with this basis. Furthermore, the Nielsen-Schreier theorem asserts that a subgroup of a free group is free, and as a consequence one can show that the hypothesis on the existence of Y is not necessary.)

Problem 3: In the situation of Problem 2, suppose that M is a right $\mathbb{Z}F$-module.

1. Show that M/MI_N becomes a right $\mathbb{Z}G$-module via
$$\overline{m}.g := \overline{m.w}$$
for $g \in G$, where $w \in F$ is a preimage, i.e., satisfies $\pi(w) = g$. (5 points)

2. If M is a free $\mathbb{Z}F$-module with a basis $B \subset M$, prove that M/MI_N is a free $\mathbb{Z}G$-module with the basis $\{\overline{b} \mid b \in B\}$. (10 points)

3. Suppose that G is finite, that F is a free group on the finite set $X \subset F$, and that N is a free group on the finite set $Y \subset N$. If M is a free $\mathbb{Z}F$-module with basis $B \subset M$, show that MI_N is a free $\mathbb{Z}F$-module with basis $\{b(y-1) \mid b \in B, y \in Y\}$. (10 points)

Problem 4: Suppose that A_0 is an abelian group and that

$$\ldots \subset A_2 \subset B_1 \subset A_1 \subset B_0 \subset A_0$$

is a descending sequence of subgroups. Show that the sequence

$$\ldots \longrightarrow A_2/A_3 \longrightarrow B_1/B_2 \longrightarrow A_1/A_2 \longrightarrow B_0/B_1 \longrightarrow A_0/A_1$$

is exact, except at the last term. (25 points)

Due date: Friday, March 26, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.