Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 7**

Homological Algebra

Problem 1: Suppose that R is a ring. A right R-module I is called injective if, for every injective module homomorphism $f : M \to N$ and every module homomorphism $g : M \to I$, there is a module homomorphism $h : N \to I$ with the property that $h \circ f = g$.

1. Draw this property as a diagram. (1 point)
2. Suppose that a right R-module I satisfies $\mathrm{Ext}^1_R(M, I) = \{0\}$ for every right R-module M. Show that I is injective. (24 points)

(Remark: The converse of the preceding assertion is also true, but you do not need to show that.)

Problem 2: Suppose that R is a ring and that $(M_i)_{i \in I}$ is a family of right R-modules, where I is a not necessarily finite index set. For each $i \in I$, let $\iota_i : M_i \to M$ be a module homomorphism to another R-module M. Suppose that, for a family $(f_i)_{i \in I}$ of module homomorphisms $f_i : M_i \to N$, there is a unique module homomorphism $f : M \to N$ so that $f \circ \iota_i = f_i$.

1. Show that each ι_i is injective. (10 points)
2. Show that
$$M \cong \bigoplus_{i \in I} M_i$$
(15 points)

Problem 3: Suppose that R is a ring and that $(M_i)_{i \in I}$ is a family of right R-modules, where I is a not necessarily finite index set. For a left R-module N, show that

$$\left(\bigoplus_{i \in I} M_i \right) \otimes_R N \cong \bigoplus_{i \in I} M_i \otimes_R N$$

(Remark: Note that $M_i \otimes_R N$ is only an abelian group, i.e., a $\mathbb{Z}$-module.) (25 points)

Problem 4: Suppose that R is a ring. A right R-module M is called flat if, for each short exact sequence

$$\{0\} \longrightarrow N \xrightarrow{f} N' \xrightarrow{g} N'' \longrightarrow \{0\}$$

of left R-modules, the sequence

$$\{0\} \longrightarrow M \otimes_R N \xrightarrow{\mathrm{id}_M \otimes_R f} M \otimes_R N' \xrightarrow{\mathrm{id}_M \otimes_R g} M \otimes_R N'' \longrightarrow \{0\}$$

is also short exact.

1. Show that R is flat as a right module over itself. (5 points)
2. Show that every free module is flat. (10 points)
3. Show that every projective module is flat. (10 points)

Due date: Wednesday, March 17, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.