Homological Algebra

Problem 1: Show that the map γ in the inflation-restriction sequence does not depend on the chosen transversal T. (25 points)

(Remark: The inflation-restriction sequence is the sequence (7.29) on page 213 of the textbook. The choice of the transversal is made on page 215 there.)

Problem 2: Suppose that G is a group and that H is a subgroup of G. For $\sigma, \tau \in G$, we denote conjugates by $\tau^\sigma := \sigma^{-1}\tau\sigma$. Let $(B_n(G))$ be the standard resolution for G and $(B_n(H^\sigma))$ be the standard resolution for H^σ. Then $B_n(H^\sigma)$ is a free abelian group with a $\mathbb{Z}$-basis consisting of the elements $[\sigma_1^\sigma|\cdots|\sigma_n^\sigma]\sigma_{n+1}^\sigma$, where $\sigma_1, \ldots, \sigma_{n+1} \in H$.
For $k = 1, \ldots, n$, we let $\eta_{k,n-1} : B_{n-1}(H^\sigma) \to B_n(G)$ be the unique group homomorphism that is given on basis elements by

$$\eta_{k,n-1}([\sigma_1^\sigma|\cdots|\sigma_{n-1}^\sigma]\sigma_n^\sigma) = [\sigma_1^\sigma|\cdots|\sigma_{k-1}^\sigma|\sigma^{-1}|\sigma_k|\cdots|\sigma_{n-1}]\sigma_n$$

Let $\omega_{n-1} := \sum_{k=1}^n (-1)^{k-1}\eta_{k,n-1}$.

1. Write out ω_1 explicitly. (1 point)
2. Write out ω_2 explicitly. (1 point)
3. Show that

$$(\partial_3 \circ \omega_2)([\sigma_1^\sigma|\sigma_2^\sigma]\sigma_3^\sigma) + (\omega_1 \circ \partial_2)([\sigma_1^\sigma|\sigma_2^\sigma]\sigma_3^\sigma) = [\sigma_1|\sigma_2]\sigma_3 - [\sigma_1^\sigma|\sigma_2^\sigma]\sigma^{-1}\sigma_3$$

(20 points)

4. Conjecture a generalisation of this formula for arbitrary n instead of 2. (3 points)

(Remark: You do not need to prove your conjecture, but it needs to reduce correctly to the special case. Try to keep it simple.)

Problem 3: In the situation of Problem 2 on Sheet 5, denote the standard resolution of G by (B_n) and let $\varphi_n : B_n \to B'_n$ be the unique Γ-linear map that takes the values

$$\varphi_n([g_1|g_2|\ldots|g_n]) = (g_1g_2\cdots g_n, g_2\cdots g_n, \ldots, g_n, e)$$

on the basis elements, where $e \in G$ is the unit element. Furthermore, let $\psi_n : B'_n \to B_n$ be the unique group homomorphism that takes the values

$$\psi_n(h_0, h_1, \ldots, h_n) = [h_0h_1^{-1}|h_1h_2^{-1}|\ldots|h_{n-1}h_n^{-1}]h_n$$

on the basis elements.

1. Show that φ_n and ψ_n are inverse mappings. (11 points)
2. For $n \in \mathbb{N}$ and $i = 0, \ldots, n$, show that $\varphi_{n-1} \circ d_{n,i} = d'_{n,i} \circ \varphi_n$, where $d_{n,i}$ was defined in Problem 1 on Sheet 4. (11 points)
3. Conclude that (B_n) and (B'_n) are isomorphic chain complexes. (3 points)

(Remark: (B'_n) is therefore also a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, called the homogeneous standard resolution.)

Problem 4: Suppose that R is a ring, that M and N are right R-modules, and that

$$\cdots \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} M$$

is a projective resolution of M. Show that the zeroth cohomology group of the cochain complex

$$\{0\} \longrightarrow \mathrm{Hom}_R(X_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(X_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_R(X_2, N) \xrightarrow{d_3^*} \cdots$$

is isomorphic to $\mathrm{Hom}_R(M, N)$. (25 points)

(Remark: Use Problem 3 on Sheet 5. Note that M is not considered part of the complex X. Recall that the cohomology groups of the above cochain complex are denoted by $\mathrm{Ext}_R^n(M, N)$, so that the problem asserts that $\mathrm{Ext}_R^0(M, N) \cong \mathrm{Hom}_R(M, N)$. In contrast to claims found in several textbooks, the groups $\mathrm{Ext}_R^n(M, N)$ obviously depend on the chosen resolution X, although the notation does not reflect this. However, as stated in Theorem 7.21 in the textbook, the comparison theorem implies that different resolutions lead to isomorphic groups $\mathrm{Ext}_R^n(M, N)$.)

Due date: Wednesday, March 10, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.