Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 5**

Homological Algebra

Problem 1: In the situation of Problem 2 on Sheet 4, we assume that $e \in T$, where e is the unit element of G. Let $(B_n(G))$ be the standard resolution for G and $(B_n(H))$ be the standard resolution for H. Recall that $B_n(G)$ is the free abelian group with basis $[g_1|\dots|g_n]g_{n+1}$ for $g_i \in G$. Let

$$\theta_n : B_n(G) \to B_n(H)$$

be the unique group homomorphism that takes on these basis elements the values

$$\theta_n([g_1|\dots|g_n]g_{n+1}) = [h_{g_2\cdots g_{n+1}.e}(g_1)|h_{g_3\cdots g_{n+1}.e}(g_2)|\dots|h_{g_n g_{n+1}.e}(g_{n-1})|h_{g_{n+1}.e}(g_n)]h_e(g_{n+1})$$

1. Show that θ_n is $\mathbb{Z}[H]$-linear. (10 points)
2. For the operators $d_{n,i}$ defined in Problem 1 on Sheet 4, show that $d_{n,i} \circ \theta_n = \theta_{n-1} \circ d_{n,i}$. (12 points)
3. Conclude that (θ_n) is a chain map from $(B_n(G))$ to $(B_n(H))$. (3 points)

Problem 2: Suppose that G is a group and that $\Gamma := \mathbb{Z}G$ is its integral group ring. For $n \in \mathbb{N}_0$, let B'_n be the free abelian group on the set G^{n+1}, whose elements are denoted $(g_0, g_1, \dots, g_n)$ in this context. For $n \in \mathbb{N}$ and $i = 0, \dots, n$, define $d'_{n,i} : B'_n \to B'_{n-1}$ as the unique group homomorphism that takes the values

$$d'_{n,i}(g_0, g_1, \dots, g_n) := (g_0, g_1, \dots, g_{i-1}, g_{i+1}, \dots, g_n)$$

on the basis elements.

1. Show that $d'_{n,i} \circ d'_{n+1,j} = d'_{n,j-1} \circ d'_{n+1,i}$ if $i < j$. (12 points)
2. For the operators $\partial'_n := \sum_{i=0}^{n} (-1)^i d'_{n,i}$, show that $\partial'_n \circ \partial'_{n+1} = 0$. (3 points)
3. Show that there is a unique right Γ-module structure on B'_n with the property that, for $g \in G$, we have
$$(g_0, g_1, \dots, g_n)g = (g_0 g, g_1 g, \dots, g_n g)$$
for all basis elements $(g_0, g_1, \dots, g_n)$ of B'_n. (3 points)
4. Show that B'_n is free with respect to this module structure. (5 points)
5. Show that $d'_{n,i}$ is Γ-linear. (2 points)

Problem 3: Suppose that R is a ring and that $M \xrightarrow{f} N \overset{g}{\twoheadrightarrow} P$ is an exact sequence of right R-modules, i.e., that g is surjective and $\ker(g) = \operatorname{im}(f)$. For another right R-module Q, show that

$$\operatorname{Hom}_R(P, Q) \overset{g^*}{\rightarrowtail} \operatorname{Hom}_R(N, Q) \xrightarrow{f^*} \operatorname{Hom}_R(M, Q)$$

is an exact sequence of abelian groups, where $f^* : \operatorname{Hom}_R(N, Q) \to \operatorname{Hom}_R(M, Q)$ denotes the precomposition map $h \mapsto h \circ f$, and g^* is defined analogously. (Note that it is claimed that g^* is injective.) (25 points)

(Remark: In category theory, this fact is stated by saying that the Hom-functor is left exact in the contravariant variable. It is also left exact in the covariant variable, as we saw in Problem 4 on Sheet 4.)

Problem 4: Suppose that R is a ring and that $M \xrightarrow{f} N \overset{g}{\twoheadrightarrow} P$ is an exact sequence of right R-modules, i.e., that g is surjective and $\ker(g) = \operatorname{im}(f)$. For a left R-module Q, show that

$$M \otimes_R Q \xrightarrow{f \otimes_R \mathrm{id}_Q} N \otimes_R Q \overset{g \otimes_R \mathrm{id}_Q}{\twoheadrightarrow} P \otimes_R Q$$

is an exact sequence of abelian groups. (Note that it is claimed that $g \otimes_R \mathrm{id}_Q$ is surjective.) (25 points)

(Remark: In category theory, this fact is stated by saying that the tensor functor is right exact in the first variable. A very similar argument shows that it is also right exact in the second variable. It is understood here that a fixed tensor product of any two modules has been chosen, for example the one arising from a certain standard construction.)

Due date: Monday, March 1, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.