Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 4**

Homological Algebra

Problem 1: Suppose that G is a group, that $\Gamma := \mathbb{Z}G$ is its integral group ring, and that $(B_n)_{n\in\mathbb{N}_0}$ is the unnormalized standard resolution. For $n \in \mathbb{N}$ and $i = 0, \ldots, n$, define $d_{n,i} : B_n \to B_{n-1}$ as the unique Γ-linear map that takes the values

$$d_{n,i}([g_1|g_2|\ldots|g_n]) := \begin{cases} [g_2|\ldots|g_n] & : i = 0 \\ [g_1|g_2|\ldots|g_i g_{i+1}|\ldots|g_n] & : i = 1, \ldots, n-1 \\ [g_1|g_2|\ldots|g_{n-1}]g_n & : i = n \end{cases}$$

on the basis elements.

1. Show that $d_{n,i} \circ d_{n+1,j} = d_{n,j-1} \circ d_{n+1,i}$ if $i < j$. (20 points)
2. Show that the boundary operators ∂_n of the standard resolution can be written in the form $\partial_n = \sum_{i=0}^{n} (-1)^i d_{n,i}$. (3 points)
3. Conclude that $\partial_n \circ \partial_{n+1} = 0$. (2 points)

Problem 2: Suppose that G is a group and that H is a subgroup of G with finite index. Let T be a system of left coset representatives, so that

$$G = \bigcup_{t\in T} tH$$

For $g \in G$ and $t \in T$, we define the elements $g.t \in T$ and $h_t(g) \in H$ by the condition that

$$gt = (g.t)h_t(g)$$

For a right G-module M, let $\varphi : H \to M$ be a derivation (cf. page 207 of the textbook) and define

$$\psi : G \to M,\ g \mapsto \psi(g) := \sum_{t\in T} \varphi(h_t(g))t^{-1}$$

Show that ψ is again a derivation. (25 points)

Problem 3: Suppose that M is a right module and that N is a left module over the ring R, and that the balanced map $t : M \times N \to T$ to the abelian group T is a tensor product of M and N in the sense of Problem 4 on Sheet 2. Show that the image of t generates T as an abelian group.
(25 points)

(Remark: This property was already stated without proof in Problem 3 on Sheet 3.)

Problem 4: Suppose that R is a ring and that $M \overset{f}{\rightarrowtail} N \overset{g}{\to} P$ is an exact sequence of right R-modules, i.e., that f is injective and $\ker(g) = \operatorname{im}(f)$. For another right R-module Q, show that

$$\operatorname{Hom}_R(Q, M) \overset{f_*}{\rightarrowtail} \operatorname{Hom}_R(Q, N) \overset{g_*}{\to} \operatorname{Hom}_R(Q, P)$$

is an exact sequence of abelian groups, where $f_* : \operatorname{Hom}_R(Q, M) \to \operatorname{Hom}_R(Q, N)$ denotes the postcomposition map $h \mapsto f \circ h$, and g_* is defined analogously. (Note that it is claimed that f_* is injective.)
(25 points)

(Remark: In category theory, this fact is stated by saying that the Hom-functor is left exact in the covariant variable. It is also left exact in the contravariant variable, in an appropriate sense.)

Due date: Wednesday, February 17, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.