Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 3**

Homological Algebra

Problem 1: Suppose that R is a ring, and that

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d_{n+1}} & X_n & \xrightarrow{d_n} & X_{n-1} & \xrightarrow{d_{n-1}} & X_{n-2} & \xrightarrow{d_{n-2}} & \cdots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & \\
\cdots & \xrightarrow{d'_{n+1}} & Y_n & \xrightarrow{d'_n} & Y_{n-1} & \xrightarrow{d'_{n-1}} & Y_{n-2} & \xrightarrow{d'_{n-2}} & \cdots
\end{array}$$

is a chain map between chain complexes of right R-modules.

1. If Z is another right R-module, show that the induced maps

$$f_n^* : \operatorname{Hom}_R(Y_n, Z) \to \operatorname{Hom}_R(X_n, Z),\ f \mapsto f_n^*(f) := f \circ f_n$$

yield a cochain map $(f_n^*)_{n\in\mathbb{Z}}$ from the cochain complex $(\operatorname{Hom}_R(Y_n, Z))_{n\in\mathbb{Z}}$ to the cochain complex $(\operatorname{Hom}_R(X_n, Z))_{n\in\mathbb{Z}}$. (10 points)

2. Suppose that the chain map $(f_n)_{n\in\mathbb{Z}}$ is homotopic to another chain map $(g_n)_{n\in\mathbb{Z}}$ of R-linear maps $g_n : X_n \to Y_n$, so that there are R-linear maps $s_n : X_n \to Y_{n+1}$ with the property that

$$f_n - g_n = d'_{n+1} \circ s_n + s_{n-1} \circ d_n$$

Show that the mappings

$$s_n^* : \operatorname{Hom}_R(Y_{n+1}, Z) \to \operatorname{Hom}_R(X_n, Z),\ f \mapsto s_n^*(f) := f \circ s_n$$

yield a cochain homotopy between the cochain maps $(f_n^*)_{n\in\mathbb{Z}}$ and $(g_n^*)_{n\in\mathbb{Z}}$. (15 points)

(This problem should be compared with the end of the proof of Theorem (7.21) on page 205 in the textbook.)

Problem 2: Suppose that R is a ring and that $(X_n)_{n\in\mathbb{N}_0}$ is a sequence of of right R-modules. Suppose that, for $n \in \mathbb{N}$ and $i = 0, \ldots, n$, we are given maps $d_{n,i} : X_n \to X_{n-1}$ with the property that

$$d_{n,i} \circ d_{n+1,j} = d_{n,j-1} \circ d_{n+1,i}$$

for $i < j$, and define

$$d_n := \sum_{i=0}^{n} (-1)^i d_{n,i}$$

Show that $d_n \circ d_{n+1} = 0$. (25 points)

Problem 3:

1. Suppose that R is a ring, that $f : M \to M'$ is a homomorphism between right R-modules, and that $g : N \to N'$ is a homomorphism between left R-modules. If $M \otimes_R N$ and $M' \otimes_R N'$ are tensor products in the sense of Problem 4 on Sheet 2, show that there is a unique group homomorphism $h : M \otimes_R N \to M' \otimes_R N'$ that satisfies $h(m \otimes_R n) = f(m) \otimes_R g(n)$. The map h is often denoted by $f \otimes_R g$ or briefly $f \otimes g$, so that this equation reads

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$$

2. For two rings R and S, an S-R-bimodule is a left S-module M that is simultaneously a right R-module in such a way that the condition $(sm)r = s(mr)$ is satisfied for all $r \in R$, $m \in M$, and $s \in S$.

   Suppose that in addition N is an R-T-bimodule, where T is a third ring. Show that a tensor product $M \otimes_R N$ has a unique structure as a S-T-bimodule in such a way that the equation

$$(s(m \otimes_R n))t = s((m \otimes_R n)t) = sm \otimes_R nt$$

   holds for all $m \in M$, $n \in N$, $s \in S$, and $t \in T$.

(Hint: If you did not show this before, you can use without proof that the so-called decomposable tensors $m \otimes_R n$ generate $M \otimes_R N$ as an abelian group.) (25 points)

Problem 4: Suppose that R, S, and T are rings.

1. If M is an S-R-bimodule and P is an S-T-bimodule, show that the space $\mathrm{Hom}_S(M, P)$ is an R-T-bimodule via

$$(rf)(m) := f(mr) \qquad \text{and} \qquad (ft)(m) := f(m)t$$

   for $r \in R$, $t \in T$, $m \in M$, and $f \in \mathrm{Hom}_S(M, P)$. (10 points)

2. If M is an S-R-bimodule, N is an R-T-bimodule, and P is a S-T-bimodule, show that the map

$$a : \mathrm{Hom}_S(M \otimes_R N, P) \to \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, P))$$

   with $(a(f)(n))(m) := f(m \otimes_R n)$ is bijective, and that it restricts to a bijection between the maps that are in addition right T-linear. (15 points)

(Remark: The map a is the so-called standard adjunction.)

Due date: Monday, February 8, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.