Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 2**

Homological Algebra

Problem 1: Suppose that P and Q are right modules over the ring R. Their (external) direct sum $P \oplus Q$ is defined as the Cartesian product of P and Q with the componentwise operations. Show that $P \oplus Q$ is projective if and only if both P and Q are projective. (25 points)

Problem 2: Suppose that P is a right module over the ring R. Show that P is projective if and only if there exists another module Q with the property that $P \oplus Q$ is a free module. (25 points)

Problem 3: Suppose that

$$H \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} F \qquad \text{and} \qquad H \xrightarrow{\iota_2} G_2 \xrightarrow{\pi_2} F$$

are two group extensions that satisfy the assumptions of Problem 3 on Sheet 1; in particular H is abelian and both extensions induce the same action of F on H described by a group homomorphism $T : F \to \mathrm{Aut}(H)$. Suppose that the two extensions are described by factor sets

$$f_1 : F \times F \to H \qquad \text{and} \qquad f_2 : F \times F \to H$$

respectively. Show that the extension $H \xrightarrow{\iota_3} G_3 \xrightarrow{\pi_3} F$ constructed in Problem 3 on Sheet 1 is described by the factor set $f_3 := f_1 + f_2$ and that the action of F on H induced by this extension is also described by T. (25 points)

(Remark: The extension G_3 therefore interprets what the sum of factor sets means from the point of view of extensions. It is known as the Baer sum of the extensions G_1 and G_2.)

Problem 4: Suppose that M is a right module and that N is a left module over the ring R. A map

$$b : M \times N \to A$$

into an abelian group A is called balanced, or middle linear, if it satisfies

1. $b(m + m', n) = b(m, n) + b(m', n)$
2. $b(m, n + n') = b(m, n) + b(m, n')$
3. $b(mr, n) = b(m, rn)$

for all elements $m, m' \in M$, $n, n' \in N$, and $r \in R$.

A balanced map $t : M \times N \to T$ is called a tensor product of M and N if it is universal in the sense that for every other balanced map $b : M \times N \to A$ into an abelian group A, there is a unique group homomorphism $f : T \to A$ such that $f \circ t = b$.
Show that, in the case where this second balanced map b is also a tensor product, the unique homomorphism f is an isomorphism. (25 points)

(Remark: There is a specific construction that shows that a tensor product with the stated properties exists. The tensor product arising from this construction is usually denoted by $M \otimes_R N$ instead of T, and one then writes $m \otimes_R n$ instead of $t(m, n)$.)

Due date: Monday, February 1, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.