Memorial University of Newfoundland **Winter Semester 2021**
Yorck Sommerhäuser **MATH 6323: Sheet 1**

Homological Algebra

Problem 1: We write the group with two elements additively as $\mathbb{Z}_2 = \{0, 1\}$, so that $1 + 1 = 0$, and define $f\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ via

$$f(0,0) = f(1,0) = f(0,1) = 0 \qquad \text{and} \qquad f(1,1) = 1$$

Show that f is a factor set, i.e., that it satisfies

$$f(i, j+k) + f(j,k) = f(i+j,k) + f(i,j)$$

for all $i, j, k \in \mathbb{Z}_2$. In other words, show that Equation (7.5) on page 193 of the textbook is satisfied if $T(i)$ is the the identity mapping for all $i \in \mathbb{Z}_2$.

(Hint: In principle, you need to verify eight equations. Try to organise your work so that you have to consider fewer cases.) (25 points)

Problem 2: Suppose that F and H are two groups and that $f\colon F \times F \to H$ and $T\colon F \to \mathrm{Aut}(H)$ are mappings that satisfy Equations (7.4) and (7.5) on page 193 of the textbook, but not necessarily Equation (7.3). Show that there is a function $h\colon F \to H$ such that $f'\colon F \times F \to H$ and $T'\colon F \to \mathrm{Aut}(H)$ defined by Equations (7.8) and (7.9) on page 195 of the textbook, i.e.,

$$T'(\sigma)(x) = h(\sigma)^{-1} T(\sigma)(x) h(\sigma)$$

and

$$f'(\sigma,\tau) = h(\sigma\tau)^{-1} f(\sigma,\tau) T(\tau)(h(\sigma)) h(\tau)$$

now satisfy Equation (7.3), i.e., satisfy

$$T'(1) = \mathrm{id}_H \qquad \text{and} \qquad f'(\sigma,1) = f'(1,\tau) = 1$$

We say that (T', f') is normalized. (25 points)

Problem 3: Suppose that

$$H \xrightarrow{\iota_1} G_1 \xrightarrow{\pi_1} F \qquad \text{and} \qquad H \xrightarrow{\iota_2} G_2 \xrightarrow{\pi_2} F$$

are two group extensions, where H is abelian. We assume that both extensions induce the same action of F on H, so that there is a group homomorphism $T : F \to \mathrm{Aut}(H)$ which satisfies

$$g_1^{-1}\iota_1(h)g_1 = \iota_1(T(\pi_1(g_1))(h)) \qquad g_2^{-1}\iota_2(h)g_2 = \iota_2(T(\pi_2(g_2))(h))$$

for all $h \in H$, $g_1 \in G_1$, and $g_2 \in G_2$. Consider the groups

$$U := \{(g_1, g_2) \in G_1 \times G_2 \mid \pi_1(g_1) = \pi_2(g_2)\}$$

and $D := \{(\iota_1(h), \iota_2(h^{-1})) \mid h \in H\}$.

1. Show that D is normal in U. (5 points)
2. For $G_3 := U/D$, define

$$\pi_3\left(\overline{(g_1, g_2)}\right) := \pi_1(g_1) \qquad \text{and} \qquad \iota_3(h) = \overline{(\iota_1(h), 1)}$$

   Show that $H \xrightarrow{\iota_3} G_3 \xrightarrow{\pi_3} F$ is again an extension. (20 points)

Problem 4: Suppose that P is a module with the property that, for every surjective module homomorphism $\pi\colon Q \to P$, there is a module homomorphism $\iota\colon P \to Q$ with $\pi \circ \iota = \mathrm{id}_P$. Show that P is projective. (25 points)

Due date: Monday, January 25, 2021. Write your solution on letter-sized paper, scan it and send it back to me via e-mail. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Similarly, prove every assertion that you make in full detail. It is not necessary to copy down the problems again.