Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 4321: Sheet 8**

Group Theory

Problem 1: In this and the following problems, p and q are distinct primes. Suppose that G is a nonabelian group of order pq^2, where $p > q$. If there is more than one p-Sylow subgroup, show that $q = 2$ and $p = 3$, so that $|G| = 12$.

Problem 2: Suppose that G is a nonabelian group of order pq^2 with a unique p-Sylow subgroup, and that any q-Sylow subgroup is isomorphic to $\mathbb{Z}_q \times \mathbb{Z}_q$.

1. Show that G is isomorphic to a semidirect product
$$G \cong \mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_q \times \mathbb{Z}_q)$$
where $\psi : \mathbb{Z}_q \times \mathbb{Z}_q \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism.
2. Show that ψ is not injective and not trivial (i.e., not constantly equal to the unit element). Therefore, its kernel has order q.
3. Show that q divides $p - 1$.
4. Find a nontrivial element in the center of $\mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_q \times \mathbb{Z}_q)$.
5. Show that $G \cong H \times \mathbb{Z}_q$, the direct product of a nonabelian group H of order pq and $\mathbb{Z}_q$.

Problem 3: Suppose that G is a nonabelian group of order pq^2 with a unique p-Sylow subgroup, and that any q-Sylow subgroup is isomorphic to $\mathbb{Z}_{q^2}$.

1. Show that G is isomorphic to a semidirect product
$$G \cong \mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{q^2}$$
where $\varphi : \mathbb{Z}_{q^2} \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism that is not trivial (i.e., not constantly equal to the unit element).
2. Show that q divides $p - 1$.
3. Suppose that $\psi : \mathbb{Z}_{q^2} \to \mathrm{Aut}(\mathbb{Z}_p)$ is another homomorphism, and that the kernel of both φ and ψ have order q. Show that the semidirect products $\mathbb{Z}_p \rtimes_\varphi \mathbb{Z}_{q^2}$ and $\mathbb{Z}_p \rtimes_\psi \mathbb{Z}_{q^2}$ are isomorphic.

Problem 4:

1. Show that there exists an injective homomorphism from $\mathbb{Z}_{q^2}$ to $\mathrm{Aut}(\mathbb{Z}_p)$ if and only if q^2 divides $p-1$. If this is the case, show that there are exactly $q(q-1)$ such injective homomorphisms.

2. If φ and ψ are injective homomorphisms from $\mathbb{Z}_{q^2}$ to $\mathrm{Aut}(\mathbb{Z}_p)$, show that the corresponding semidirect products

$$\mathbb{Z}_{q^2} \ltimes_{\varphi} \mathbb{Z}_p \qquad \text{and} \qquad \mathbb{Z}_{q^2} \ltimes_{\psi} \mathbb{Z}_p$$

are isomorphic.

3. Show that these two isomorphic semidirect products are not isomorphic to the semidirect product considered in the second part of the preceding problem.

Due date: There is no due date. The completion of this sheet is voluntary. The solutions will not be collected and will not be graded. However, these exercises provide valuable practice for the final exam.