Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 4321: Sheet 7**

Group Theory

Problem 1: For a natural number n, recall that we have defined the dihedral group D_{2n} as the semidirect product $D_{2n} := \mathbb{Z}_n \rtimes_\psi \mathbb{Z}_2$, where $\psi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$ is the group homomorphism that maps $\bar{0}$ to the identity and $\bar{1}$ to the inversion mapping, so that we have

$$\psi(\bar{1})(\bar{k}) = -\bar{k}$$

1. Find a group G that is generated by elements x and h that satisfy $x^2 = 1$, $h^n = 1$, and $xh = h^{-1}x$, but is not isomorphic to D_{2n}. (5 points)
2. Show that $G \cong D_{2n}$ if in addition $|G| = 2n$. (20 points)

Problem 2: If n is odd, show that $D_{4n} \cong D_{2n} \times \mathbb{Z}_2$. (25 points)

Problem 3: For a natural number n, we have considered in class the semidirect product $\mathbb{Z}_{2n} \rtimes_\psi \mathbb{Z}_4$, where $\psi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_{2n})$ is the unique group homomorphism that maps $\bar{1}$ to the inversion mapping. In this semidirect product, the element $(\bar{n}, \bar{2})$ has order 2, and we have defined the dicyclic group DC_{4n} as the quotient of $\mathbb{Z}_{2n} \rtimes_\psi \mathbb{Z}_4$ by the subgroup of order 2 generated by this element. We have seen that for the cosets h and x of $(\bar{1}, \bar{0})$ and $(\bar{0}, \bar{1})$, respectively, we have

$$h^{2n} = 1 \qquad h^n = x^2 \qquad hx = xh^{-1}$$

and that the order of DC_{4n} is $4n$.

1. If n is odd, show that DC_{4n} is isomorphic to the semidirect product $\mathbb{Z}_n \rtimes_\varphi \mathbb{Z}_4$ where $\varphi : \mathbb{Z}_4 \to \mathrm{Aut}(\mathbb{Z}_n)$ is the unique group homomorphism that maps $\bar{1}$ to the inversion mapping. (12 points)
2. Show that this is not the case if n is even. (13 points)

Problem 4: Suppose that G is a nonabelian group of order $4p$, where $p \geq 5$ is a prime.

1. Show that there is a unique p-Sylow subgroup. (5 points)
2. Suppose that the 2-Sylow subgroup of G is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. Show that G is isomorphic to a semidirect product

$$G \cong \mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$$

   where $\psi : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_p)$ is a group homomorphism. (5 points)
3. Show that ψ is not injective and not trivial (i.e., not constantly equal to the unit element). Therefore, its kernel has order 2. (5 points)
4. Find a nontrivial element in the center of $\mathbb{Z}_p \rtimes_\psi (\mathbb{Z}_2 \times \mathbb{Z}_2)$. (5 points)
5. Show that $G \cong D_{4p}$. (5 points)

Due date: Monday, March 27, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.