Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 4321: Sheet 6**

Group Theory

Problem 1: Suppose that N and H are groups and that $\psi : H \to \operatorname{Aut}(N)$ is a group homomorphism. We have seen in class that

$$\varepsilon_N : N \to N \rtimes_\psi H, n \mapsto (n, 1) \qquad \text{and} \qquad \varepsilon_H : H \to N \rtimes_\psi H, h \mapsto (1, h)$$

are group homomorphisms into the associated semidirect product $N \rtimes_\psi H$.

1. Show that $\varepsilon_H(h)\varepsilon_N(n)\varepsilon_H(h)^{-1} = \varepsilon_N(\psi(h)(n))$. (5 points)

2. Suppose that $f_N : N \to G$ and $f_H : H \to G$ are group homomorphisms into another group G that satisfy

   $$f_H(h)f_N(n)f_H(h)^{-1} = f_N(\psi(h)(n))$$

   Show that there is a unique group homomorphism $f : N \rtimes_\psi H \to G$ satisfying $f \circ \varepsilon_N = f_N$ and $f \circ \varepsilon_H = f_H$. (20 points)

(Remark: This is the so-called universal property of the semidirect product.)

Problem 2:

1. Consider the Klein four-group V. Show that the mapping $f : V \to V$ that maps the identity to itself and satisfies

   $$f((1, 2)(3, 4)) = (1, 4)(2, 3) \qquad f((1, 3)(2, 4)) = (1, 2)(3, 4)$$

   and $f((1, 4)(2, 3)) = (1, 3)(2, 4)$ is a group homomorphism of order 3. (5 points)

2. Suppose that $C = \{1, c, c^2\}$ is a cyclic group of order 3, and consider the group homomorphism $\varphi : C \to \operatorname{Aut}(V)$ that satisfies $\varphi(c) = f$. Show that the semidirect product $V \rtimes_\varphi C$ is isomorphic to A_4. (20 points)

Problem 3: Suppose that p and q are primes and that p divides $q - 1$. For two monomorphisms $\varphi : C_p \to \operatorname{Aut}(C_q)$ and $\psi : C_p \to \operatorname{Aut}(C_q)$, show that the semidirect products

$$C_q \rtimes_\varphi C_p \qquad \text{and} \qquad C_q \rtimes_\psi C_p$$

are isomorphic. Here C_p and C_q denote cyclic groups of order p and q, respectively. (30 points)

Problem 4: A finite abelian p-group is called elementary abelian if every non-identity element has order p.

1. Write down all elements of $\mathrm{Aut}(C_{24})$ explicitly by stating what the image of a generator of C_{24} under every automorphism is. (5 points)

2. Show that $\mathrm{Aut}(C_{24})$ is an elementary abelian 2-group. (15 points)

Due date: Monday, March 20, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.