Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 4321: Sheet 5**

Group Theory

Problem 1: Suppose that p is a prime and that G is a group of order p^2. Show that G is isomorphic to $\mathbb{Z}_{p^2}$ or to $\mathbb{Z}_p \times \mathbb{Z}_p$.
(Hint: Show first that G is abelian. For this, Problem 2 on Sheet 4 is helpful.)
(25 points)

Problem 2: Suppose that G and H are groups and that $\psi : H \to \mathrm{Aut}(G)$ is a group homomorphism. On the Cartesian product $G \times H$, we define the multiplication

$$(g, h)(g', h') := (g\ \psi(h)(g'), hh')$$

1. Show that this multiplication is associative. (15 points)
2. Show that $(1_G, 1_H)$ is a two-sided unit element for this multiplication. (2 points)
3. For $(g, h) \in G \times H$, show that $(\psi(h^{-1})(g^{-1}), h^{-1})$ is a two-sided inverse with respect to this multiplication. (8 points)

(Remark: $G \times H$ is therefore a group with respect to this multiplication. This group is called the (external) semidirect product of G and H and is denoted by $G \rtimes_\psi H$ or briefly $G \rtimes H$.)

Problem 3: Suppose that G is a group. Recall that the (short) commutator of two elements g and h of G is

$$[g, h] := ghg^{-1}h^{-1}$$

Define the long commutator of elements $g_1, g_2, \ldots, g_n \in G$ as the element

$$g_1 \cdot g_2 \cdots g_n \cdot g_1^{-1} \cdot g_2^{-1} \cdots g_n^{-1}$$

Show that the product of an arbitrary number of (short) commutators can be written as a long commutator.
(Hint: Try two short commutators first. Then use induction on the number of short commutators.) (25 points)

Problem 4: The second proof Cauchy's theorem given in class is due to James H. McKay. It was published in 1959 in the 'American Mathematical Monthly', Volume 66, on page 119.

1. Obtain a copy of this article, print it, and submit it. (10 points)
2. This article was reviewed by R. G. Stanton for the 'Mathematical Reviews'. These reviews are stored in a database called MathSciNet, to which you have access through the university library. Obtain a copy of this review, print it, and submit it. (15 points)

(Hint: The review on MathSciNet has a direct link to the original article.)

Due date: Monday, March 13, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.