Memorial University of Newfoundland **Winter Semester 2017**
Yorck Sommerhäuser **MATH 4321: Sheet 4**

Group Theory

Problem 1:

1. State the Chinese Remainder Theorem for ordinary integer congruences by quoting from a textbook of your choice. Cite precisely the textbook that you use, together with the section number and the page number of the Chinese Remainder Theorem. (3 points)
2. Use the above version of the Chinese Remainder Theorem to show that $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ for relatively prime natural numbers m and n. (You need to construct a map, show that it is well-defined, show that it is injective and surjective, and that it is a group homomorphism.) (20 points)
3. For the case $m = 2$ and $n = 3$, make a table that lists explicitly the images of all six elements of $\mathbb{Z}_6$ in $\mathbb{Z}_2 \times \mathbb{Z}_3$. (2 points)

Problem 2: For a group G, the center is the set

$$Z(G) := \{g \in G \mid gh = hg \text{ for all } h \in G\}$$

of elements that commute with every element of G.

1. Show that $Z(G)$ is a characteristic subgroup of G. (5 points)
2. Show that G is abelian if $G/Z(G)$ is cyclic. In other words, in this situation we even have $G = Z(G)$. (20 points)

Problem 3: In the symmetric group $G = S_4$ on four letters, let H be the subgroup generated by the cycle $(1, 2, 3)$. Let N be the normalizer of H. Find $|N|$ and list explicitly all elements of N. Prove in detail that there are no other elements. (25 points)

Problem 4: A proper subgroup H of a group G is called maximal if there are no subgroups that are strictly larger than H and simultaneously strictly smaller than G. Every finite group obviously contains at least one maximal subgroup, often many.

Now let G be a finite cyclic group of order n.

1. Show that a subgroup H of G is maximal if and only if the index $[G : H]$ is a prime. (15 points)
2. Show that G contains a unique maximal subgroup if and only if n is a prime power, i.e., the power of a prime. (10 points)

Due date: Monday, February 13, 2017. Write your solution on letter-sized paper, and write your name on your solution. Write down all necessary computations in full detail, and explain your computations in English, using complete sentences. Prove every assertion that you make in full detail. It is not necessary to copy down the problems again or to submit this sheet with your solution.

Change of syllabus: The midterm exam will take place on Friday, February 17, not on Wednesday, February 15.