Introduction to Complex Analysis

Problem 1: Recall from class that for $z \in \mathbb{C}$ the basic trigonometric functions are defined as

$$\cos(z) := \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) := \frac{e^{iz} - e^{-iz}}{2i}$$

1. Show that $\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$. (8 points)
2. Show that $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)$. (8 points)

Problem 2: As in the real case, the remaining trigonometric functions are defined as

$$\tan(z) := \frac{\sin(z)}{\cos(z)} \qquad \cot(z) := \frac{\cos(z)}{\sin(z)} \qquad \sec(z) := \frac{1}{\cos(z)} \qquad \csc(z) := \frac{1}{\sin(z)}$$

for those $z \in \mathbb{C}$ for which the denominator is not zero. If C is a circle of radius $1/2$ around the point $1 + i$, find

$$\int_C \cot(z) dz$$

(Hint: Use the discussion of the complex zeros of sine and cosine given in Chapter 3 of the textbook.) (17 points)

Problem 3: If C is a circle of radius 3 around the origin, find

$$\int_C \frac{e^{-2z^2}}{(z - 2i)(z - 5)} dz$$

(17 points)

Problem 4: Suppose that $f : D \to \mathbb{C}$ is holomorphic in the domain D. Furthermore, suppose that D contains the circle C of radius r around z_0 and also all the points in the interior of this circle. If we have

$$|f(z_0)| < |f(z)|$$

for all $z \in C$, show that f has a zero inside the circle; i.e., that there exists z_1 inside the circle with $f(z_1) = 0$. (17 points)

Problem 5: A holomorphic function that is defined in the entire Argand plane is called an entire function. Show that there is no nonzero entire function f that satisfies $f(0) = 0$ and $|f(z)| \leq 1$ for all $z \in \mathbb{C}$ with $|z| \geq 1$. (16 points)

Problem 6: Suppose that

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a polynomial of degree n. Show that for every $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$, there exists $R > 0$ such that

$$(1 - \varepsilon)|a_n||z^n| \leq |P(z)| \leq (1 + \varepsilon)|a_n||z^n|$$

for all $z \in \mathbb{C}$ satisfying $|z| > R$. (17 points)

Due date: Thursday, November 12, 2015. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.