Memorial University of Newfoundland **Fall Semester 2015**
Yorck Sommerhäuser **MATH 3210: Sheet 6**

Introduction to Complex Analysis

Problem 1: Recall that the complex power function

$$z^c := \exp(c \log(z))$$

is 'multi-valued' in general. Find all possible values of $i^{1/3}$ according to this definition, i.e., find all possible values of z^c for $z = i$ and $c = 1/3$. Make a table that lists the real and the imaginary parts of these values explicitly. Discuss in detail how this definition relates to the definition of the complex roots $\sqrt[3]{i}$ introduced at the beginning of the semester. (17 points)

Problem 2:

1. Find the principal value of $(-i)^i$. (8 points)
2. Find the value of $(-i)^i$ when the branch of the logarithm with a branch cut along the positive real axis is used. (8 points)

Problem 3: Without computing the integral explicitly, use the standard estimate to show that the contour integral

$$\int_C \frac{z^3 + z + 2}{z^7 + z^3 + 1} dz$$

where C is a quarter-circle of radius 2 around the origin in the first quadrant, cannot be larger than $12\pi/119$. (17 points)

Problem 4: Let C be the ellipse given by the equation

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

oriented positively, i.e., counterclockwise. Find the contour integral

$$\int_C \frac{1}{z} dz$$

(17 points)

Problem 5: Let $D \subset \mathbb{C}$ be a domain in the Argand plane, and $f : D \to \mathbb{C}$ be a continuous function. If $\gamma : [c, d] \to D \cap \mathbb{R}$ is a contour that does not leave the real line, show that

$$\int_\gamma f(z)dz = \int_a^b f(x)dx$$

where $a = \gamma(c)$ and $b = \gamma(d)$. In other words, show that the contour integral and the Riemann integral from Calculus II give the same result in this situation. (17 points)

Problem 6: Let $f(z) = z^{1/2}$.

1. Explain what this means when we want to understand $f(z)$ as the principal value. (2 points)
2. For $0 < \alpha < \pi$, let $z_1 := 2e^{-i\alpha}$ and $z_2 := \bar{z}_1 = 2e^{i\alpha}$. Find

$$\int_{z_1}^{z_2} f(z)dz$$

   and discuss what this expression means. (10 points)
3. Can we set $\alpha = \pi$ in the preceding part? (2 points)
4. Does f have an antiderivative? How does this question relate to Part 2 and Part 3 of this question? (2 points)

Due date: Tuesday, November 3, 2015. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.