Memorial University of Newfoundland **Fall Semester 2015**
Yorck Sommerhäuser **MATH 3210: Sheet 3**

Introduction to Complex Analysis

Problem 1: Give an algebraic proof of the triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

for $z_1, z_2 \in \mathbb{C}$ using the following steps:

1. Show that $\mathrm{Re}(z) \leq |z|$ for all $z \in \mathbb{C}$. (6 points)
2. Show that $(z_1 + z_2)\overline{(z_1 + z_2)} \leq (|z_1| + |z_2|)^2$. (7 points)
3. Show that $|z_1 + z_2| \leq |z_1| + |z_2|$. (3 points)

Problem 2: Find

$$\lim_{z\to\infty} \frac{z^2 - i}{z^2 + i} \qquad \lim_{z\to\infty} \frac{z^3 - i}{z^2 + i} \qquad \lim_{z\to 3i} \frac{z^2 + 4z + 4}{z^2 + 2z - 3iz - 6i}$$

(As always, fully justify your answer.) (6 points each)

Problem 3: Suppose that $\lim_{z\to z_0} f(z) = 0$ and that the function $g(z)$ is bounded in a neighbourhood U of z_0 in the sense that there is a constant M such that $|g(z)| \leq M$ for all $z \in U$. Using the ε–δ–definition of a limit, show that $\lim_{z\to z_0} f(z)g(z) = 0$. (16 points)

Problem 4:

1. From the definition, show that $f(z) = \frac{1}{z}$ has the (complex) derivative $f'(z) = -\frac{1}{z^2}$. (8 points)
2. Assuming that you know that the (complex) derivative of $f(z) = z^n$ is $f'(z) = nz^{n-1}$ for positive integers n, use the first part of this problem and the chain rule to prove this formula for negative integers n.(8 points)

Problem 5: Consider the function $f(z) = \bar{z}$.

1. Use the definition to show that $f(z)$ is nowhere (complex) differentiable. (9 points)
2. Use the Cauchy-Riemann equations to show that $f(z)$ is nowhere (complex) differentiable. (8 points)

Problem 6: For the function

$$f(z) := \begin{cases} \frac{\bar{z}^2}{z} & : z \neq 0 \\ 0 & : z = 0 \end{cases}$$

decide whether $f'(0)$ exists. (17 points)

Due date: Tuesday, October 6, 2015. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.