Memorial University of Newfoundland **Fall Semester 2015**
Yorck Sommerhäuser **MATH 3210: Sheet 10**

Introduction to Complex Analysis

Problem 1: Find the Maclaurin series for the function

$$f(z) = \frac{1}{(z+1)^3}$$

in closed form, and determine its radius of convergence.
(Hint: Use differentiation of power series, starting from a variant of the geometric series.) (17 points)

Problem 2: Find the Maclaurin series for the function

$$f(z) = \mathrm{Log}(1+z)$$

in closed form, and determine its radius of convergence.
(Hint: Use integration of power series, starting from a variant of the geometric series. The symbol Log denotes the principal branch of the logarithm function.) (16 points)

Problem 3: In the Cauchy product

$$e^z \sin(z) = \sum_{n=0}^{\infty} c_n z^n$$

find the coefficients c_0, c_1, c_2, and c_3 by using multiplication of series, and determine the radius of convergence of the Cauchy product. (17 points)

Problem 4:

1. Show that the function $g(z) := \frac{\sin(z)}{z}$ is holomorphic at the origin (and also everywhere else) and find its Maclaurin series in closed form. (5 points)
2. If $f(z) = \cos(z)$, find the coefficients a_0, a_1, a_2, a_3, a_4, a_5, and a_6 in the Maclaurin series
$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} a_n z^n$$
by using division of power series, and find its radius of convergence. (7 points)

3. For the Laurent series
$$\cot(z) = \sum_{n=-\infty}^{\infty} c_n z^n$$
of the cotangent function about the origin, find the coefficients c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, c_4, and c_5, and determine where the Laurent series converges. (5 points)

(Hint: The Maclaurin series for sine and cosine are considered as known; you can use them without further justification. A derivation is given in Example 3 and Example 4 of Section 64 of the textbook.)

Problem 5: If C is a circle of radius 3 around the origin, oriented counterclockwise, find
$$\int_C \cot(z)dz$$
(17 points)

Problem 6:

1. Show that the function $f(z) := \frac{e^z-1}{z}$ is holomorphic at the origin (and also everywhere else) and find its Maclaurin series in closed form. (6 points)

2. If $g(z) := 1/f(z)$, the Bernoulli numbers are the numbers $B_n := g^{(n)}(0)$. They therefore appear in the Maclaurin series
$$g(z) = \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$
Find the Bernoulli numbers B_0, B_1, B_2, B_3, B_4, B_5, and B_6 by using division of power series. (10 points)

(Hint: The Maclaurin series for the exponential function is considered as known; you can use it without further justification. A derivation is given in Example 2 of Section 64 of the textbook; we also did this in class. The Bernoulli numbers can be used to give a closed form for the Laurent series of the cotangent function, of which we have computed only some of the coefficients in Problem 4.)

Due date: Tuesday, December 1, 2015. Please write your solution on letter-sized paper, and write your name on your solution. Please give all computations in full detail, and explain your computations in English, using complete sentences. It is not necessary to copy down the problems again or to submit this sheet with your solution.