Tufts University
Department of Mathematics
Math 87 Homework 2

Due: Thursday, September 26, at 10:30 a.m. (in class).

From the course webpage, download the four linked m-files: `bisection.m`, `secant.m`, `newton.m`. and `halley.m`. Each of these implements a function for finding the roots of a nonlinear equation, $f(x) = 0$, under different assumptions. The methods are:

Bisection: The bisection algorithm assumes that you know two points, x_ℓ and x_r, for which $f(x_\ell)$ and $f(x_r)$ have opposite signs. For continuous f, this implies there is a zero between x_ℓ and x_r. To find it, the algorithm iteratively divides the interval from x_ℓ to x_r into two parts by examining the sign of $f((x_\ell + x_r)/2)$ and discarding the endpoint of the same sign. No assumption on f is made aside from continuity, and convergence is always guaranteed. Only $f(x)$ is needed for implementation.

Secant Method: As described in class, the secant method approximates $f(x)$ by the secant line between two points, x_{k-1} and x_k. Convergence is guaranteed if the initial guesses, x_0 and x_1 are close enough to the root, and if f is twice continuously differentiable near the root with nonzero first derivative at the root. Despite these theoretical limits, only $f(x)$ is needed for implementation.

Newton's Method: As described in class, Newton's method approximates $f(x)$ by the tangent line at the previous iterate, x_{k-1}. Convergence is guaranteed if the initial guess, x_0, is close enough to the root, and if f is continuously differentiable near the root. Both $f(x)$ and $f'(x)$ are required for implementation.

Halley's Method: Halley's method (named after Edmond Halley, just as Halley's comet) approximates $f(x)$ by a ratio of linear functions determined by the previous iterate. The update formula is given by

$$x_k = x_{k-1} - \frac{2f(x_{k-1})f'(x_{k-1})}{2\left(f'(x_{k-1})\right)^2 - f(x_{k-1})f''(x_{k-1})}.$$

Convergence is guaranteed if the initial guess, x_0, is close enough to the root, and if f is three times continuously differentiable near the root. All of $f(x)$, $f'(x)$, and $f''(x)$ are required for implementation.

In general, we expect Halley's method to converge in the fewest number of iterations, then Newton's method, then the secant method, then bisection. In contrast, bisection works with the least amount of information and assumptions, then the secant method, then Newton's method, then Halley's method. In this assignment, we'll compare their performance in cases that both fit and violate the theory for these methods.

1. (30 points) The Lagrange points are the points in an orbital configuration of two bodies (the Sun and the Earth, for example) where the imbalance in gravitational forces from the two bodies provides the centripetal force needed for an object to rotate in conjunction with them. Three Lagrange points lie in the line between the two objects. Taking M_1 to be the mass of the Sun, M_2 to be the mass of the Earth, and R to be the distance between the two objects,

the distance, r, from the L_1 point, between the Earth and Sun on the line that connects them, to the Earth is given by

$$\frac{M_1}{(R-r)^2} = \frac{M_2}{r^2} + \frac{M_1}{R^2} - \frac{M_1 + M_2}{R^3} r.$$

With the same notation, the distance from the L_2 point, outside the Earth's orbit on the line between the Earth and Sun on the line that connects them, to the Earth is given by

$$\frac{M_1}{(R+r)^2} + \frac{M_2}{r^2} = \frac{M_1}{R^2} + \frac{M_1 + M_2}{R^3} r.$$

Take $M_1 = 1.9891 \times 10^{30}$kg, $M_2 = 5.9722 \times 10^{24}$kg, and $R = 1.496 \times 10^8$km, as the masses of the Sun and Earth, and their orbital distance, respectively. Compute the distances from Earth to the L_1 and L_2 points, using all of the methods outlined above. Report the number of iterations needed to find the solution for each method for each Lagrange point. Discuss how you chose your initial guesses. Note: there are many ways to define the functions needed here. Be sure your choice has a reasonable derivative near the solution, or the stopping criterion in the iterations will be difficult to satisfy.

2. (20 points) One of the uses of Newton's method is in implementing division on microprocessors where only addition and multiplication are available as primitive operations. To compute $x = a/b$, first the root of $f(x) = 1/x - b$ is found using Newton's method, then the fraction is computed with one last multiplication by a. Find the Newton's method iteration to solve $f(x) = 0$ and explain why it is well-suited to this purpose. Apply it to computing $1/b$, where b is the last 3 digits of your student number, and where b is the area code of your phone number. For these experiments, report the iterates until the approximation is consistent to 10 digits (you will need to modify the given code in order to do so). Explain why Halley's method cannot be used for this purpose.

3. (20 points) Another use for Newton's method and other rootfinding approaches is for computing square roots, by solving for zeros of $f(x) = x^2 - a$. Apply all of the above methods to computing $\sqrt{a}$ where a is the last 3 digits of your student number, and where a is the area code of your phone number. Explain how you choose good initial guesses, and report the number of iterations needed for convergence.

4. (30 points) In this problem, we'll look at performance of various methods for functions where the assumptions needed for convergence are violated. Consider the 3 functions

$$f_1(x) = x^3$$
$$f_2(x) = x^3 + 2x$$
$$f_3(x) = x^3 + 2x^{5/4}$$

Report on the iteration counts for each algorithm for each function to find the root at $x = 0$ for consistent initial guesses (that you must choose - be sure to avoid cases where the algorithms get "lucky" and find the solution $x = 0$ at their first step). Explain how this correlates with the explanations given above.