Math 87: Mathematical Modeling and Computing

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What's that?

Mathematical Modeling

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That *something* could be

- The maximum profit to be made from selling an asset
- The minimum labor cost for performing a given task
- The minimum time needed to complete a multi-step task
- The maximum amount of traffic flow through a network
- The ranking of webpages and sports teams
- The waiting times of customers in a queue
- The fluctuation of populations over time

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We'll study all of these in this class

More Complicated Models

Mathematical Modeling is at the center of many areas of science and engineering

- Weather Forecasting
- Racecar design
- Industrial processes
- Plate Tectonics
- Oil exploration
- Sub-atomic physics
- Structural deformation
- Virtual surgery

Weather Forecasting

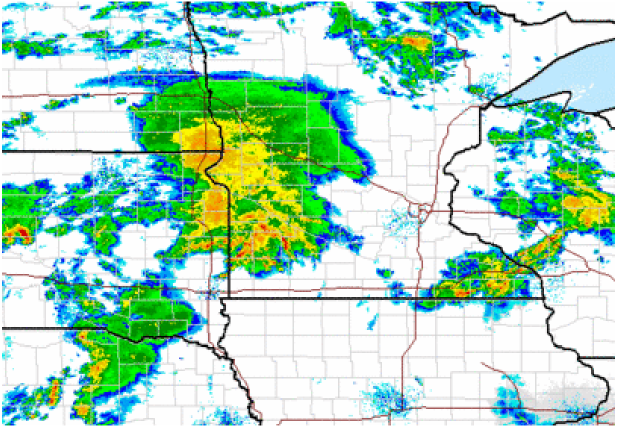


Figure from NWS

Faster Racecars

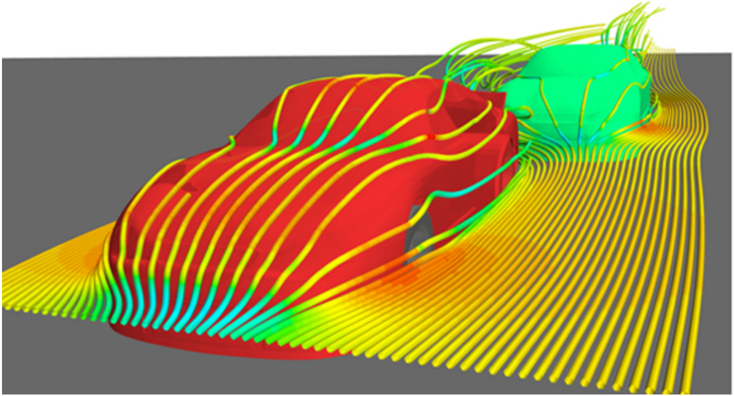


Figure from www.cd-adapco.com

New Planes

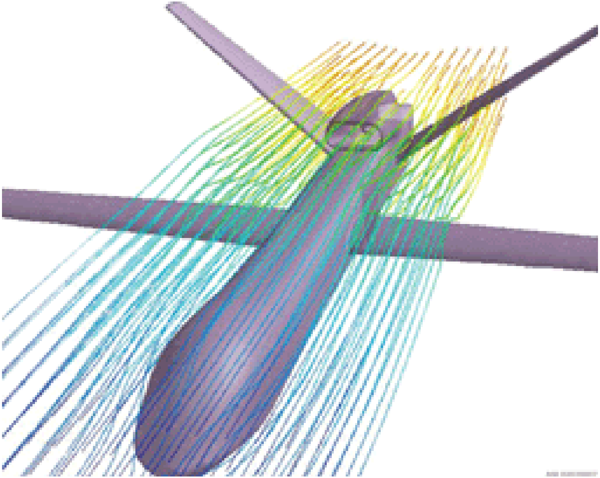
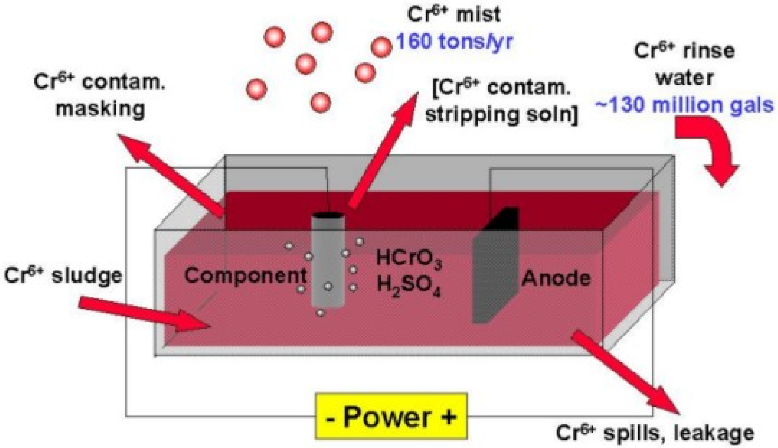


Figure from www.fluent.com

Industrial Processes



Figures from www.rowantechnology.com and www.hummerproducts.com

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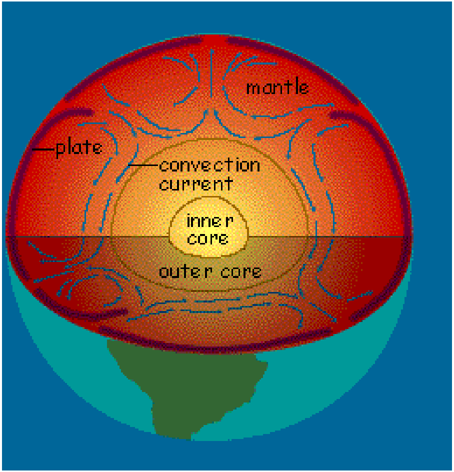
Plate Tectonics



Figures from PBS and USGS

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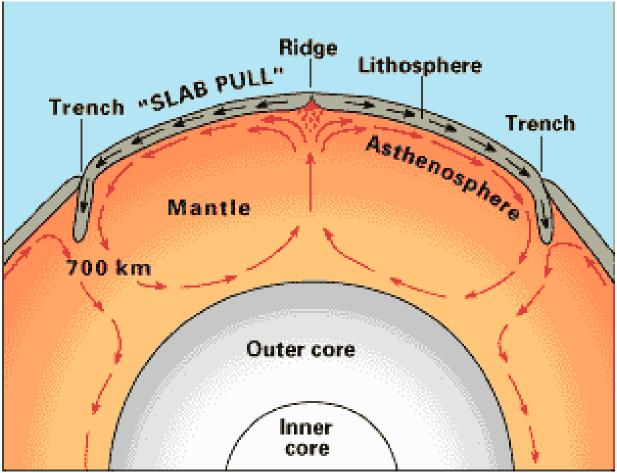

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Plate Tectonics

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Global Simulation

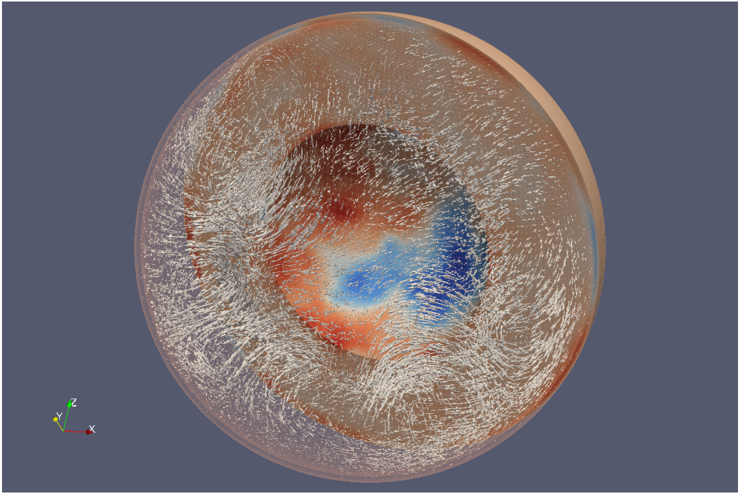

Figure courtesy T. Geenen

Global Simulation

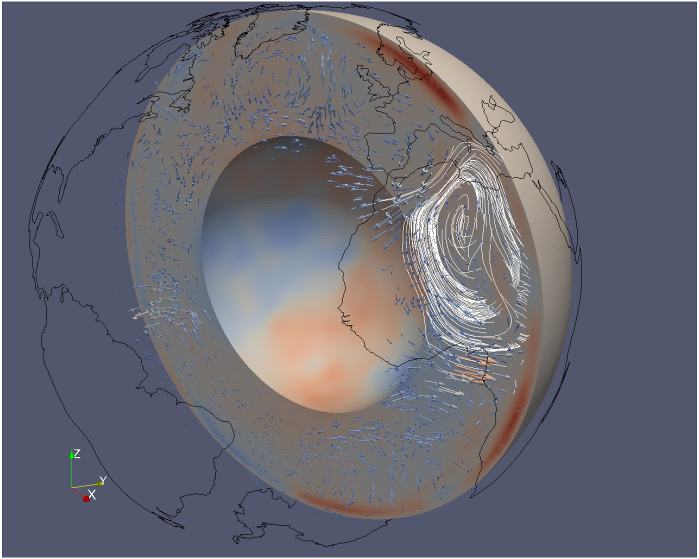

Figure courtesy T. Geenen

In this class...

We have two goals:

1. Learn how to formulate useful mathematical models
    - Formulation is very problem-dependent
    - We will focus on broad concepts and tools
    - We will look at examples in class, on homework, and in projects

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1. Learn how to formulate useful mathematical models
    - Formulation is very problem-dependent
    - We will focus on broad concepts and tools
    - We will look at examples in class, on homework, and in projects
2. Learn how to solve the models to make useful conclusions
    - Writing down the model isn't enough
    - Use model to design strategies and make conclusions
    - Recognize when pencil-and-paper can't help

Formulating Models

"essentially, all models are wrong, but some are useful"

– George Box

Looking for **predictive models**

- include all *relevant* effects
  - need to find out which are relevant
- look for simplicity/elegance where possible
  - convoluted terms are difficult to debug
- need to understand predictions and limitations
  - how sensitive are predictions to parameters?
  - are your assumptions valid in regime of interest?

Where does Computing come in?

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Basic Truth of Modeling: Most interesting/useful models can't be efficiently solved by pencil-and-paper calculation

You could have guessed that!

- In Calculus, can only integrate functions that come in certain forms
- In Differential Equations, can only solve certain types of equations
- In Linear Algebra, solving systems larger than 4×4 or 5×5 is difficult (and easy to screw up)

Our Approach to Computing

Focus first on Mathematical Modeling

- Get it “right”
- Identify computational problem to be solved

Then consider Computing

- Computer is a tool to avoid rote pencil-and-paper calculation
- Develop computing skills as we need them

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Then reconsider Mathematical Modeling

- Does computed solution make sense? Is it reasonable?
- Do we need to update our model and recompute?

Things we'll study

- Optimization, Sensitivity Analysis
- Linear Programming, Integer Programming
- Graph Algorithms, Critical Paths, Maximum Flow
- Probabilistic Modeling, Markov Chains
- Monte Carlo Methods, Queuing Theory
- Difference Equations, Population Dynamics

Optimization

Continuous optimization problems

- Take derivatives, set to zero

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What's new?

- Optimizing over multiple variables (Math 13/42)
- Sensitivity to parameters
- Approximate differentiation

Linear Programming

Another technique of optimization

- Maximize/Minimize known linear function
- Constraints given by inequalities

Many applications

- Inequality constraints are natural in real world
  - Represent limited resources, or required bounds on solution

Special case of Integer Programming

- Non-integer answers don't always make sense!
  - Optimal class size can't have fractional students

Graph Algorithms

Many scheduling and flow problems are naturally described by graphs

- Nodes describe tasks to be done
- Edges describe dependencies between tasks

Critical Path Analysis identifies both time needed and bottlenecks

- Which tasks in a complex schedule constrain the timeline?
- Which tasks are flexible in scheduling?

Maximum Flow problems identify bottlenecks in networks

- Which highways or bridges slow traffic the most?
- How should computer/telephone networks be expanded to improve bandwidth?

Markov Chains

Markov Chains model probabilistic transitions

- What percentage of one group move into another group in a given timeframe?
  - How many voters switch affiliation every two/four years?
  - How many consumers change brands each year?

Also many other applications

- In sports leagues with limited interleague play, how do you rank teams that don't directly play one-another?
- In complicated graphs, like those of webpages on a given topic, which nodes are more important than others?

We'll study

- The steady-state behavior of Markov processes
- Convergence to steady state.

Monte Carlo and Queuing

How do you flip a coin on a computer?

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- Generate a random number

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For processes controlled by random events with known probabilities

- Generate random samples
- Average over many independent samples

This naturally models queue-like operations

- new entries are added with given probability in time
- old entries are removed with given probability in time

Difference Equations

Natural models of populations from generation to generation

- If interactions are linear and homogeneous, reduce to Markov Chains
- Nonlinear and non-homogeneous interactions model more complex behavior
- Examples include
  - Fibonacci numbers, early model of rabbit population
  - Logistic map, model of population with limited resources
  - Predator-prey or host-parasite interactions
  - Spread of disease in a population
- Closely related to solution of differential equations

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Mathematics:

- Optimization (continuous and discrete)
- Linear algebra
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- Probability
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Computing:

- Basic matlab syntax
- Graphing and visualization
- Loops, control statements
- Linear algebra