Tufts University
Department of Mathematics
Math 250-03 Homework 6

Due: Thursday, November 8, at 3:00 p.m. (in class).

1. (20 points) Prove the following Poincaré-Friedrichs Theorem.

   Define $H_1([a,b]^2) = \{u : [a,b]^2 \to \mathbb{R} \mid u, u_x, u_y \in L_2([a,b]^2)\}$, where $L_2([a,b]^2)$ is the space consisting of limits of converging sequences of continuous functions on $[a,b]^2$, just as in 1D. Further, define

   $$H_{1,0}([a,b]^2) = \{u \in H_1([a,b]^2) \mid u(a,y) = u(b,y) = u(x,a) = u(x,b) = 0 \text{ for } a \le x, y \le b\}.$$

   Prove that $\frac{2}{(b-a)^2}\|u\|^2 \le \|u_x\|^2 + \|u_y\|^2$ for $u \in H_{1,0}([a,b]^2)$, where the norm is the $L_2([a,b]^2)$ norm, $\|u\|^2 = \int_a^b \int_a^b u^2(x,y)\,dy\,dx$. *Hint:* First show that $\|u\|^2 \le (b-a)^2\|u_x\|^2$ and $\|u\|^2 \le (b-a)^2\|u_y\|^2$, following the proof done in class.

2. (20 points) Write the weak form of the PDE

   $$\begin{cases} -u_{xx} - u_{yy} = f(x,y) & a < x, y < b \\ u(x,a) = u(x,b) = 0 & a \le x \le b \\ u(a,y) = u(b,y) = 0 & a \le y \le b \end{cases}.$$

   Show that the bilinear form that you get is continuous and coercive in $H_{1,0}([a,b]^2)$ with norm

   $$\|u\|_1^2 = \|u\|_0^2 + \|u_x\|_0^2 + \|u_y\|_0^2,$$

   where $\|u\|_0^2 = \int_a^b \int_a^b u^2(x,y)\,dy\,dx$ is the norm on $L_2([a,b]^2)$. What else do you need to know to conclude that the PDE has a weak solution in $H_{1,0}([a,b]^2)$?

3. (10 points) Show that the bilinear form given by $a(u,v) = \int_a^b \int_a^b K(x,y)(u_x v_x + u_y v_y)\,dy\,dx$ is continuous and coercive in $H_{1,0}([a,b]^2)$ when $0 < K_0 \le K(x,y) \le K_1 < \infty$ for all x, y.

4. (10 points) Let $\{x_n\} \subset \mathcal{H}$, $x \in \mathcal{H}$, for Hilbert space $\mathcal{H}$. Show that $x_n \to x$ strongly if and only if $x_n \to x$ weakly and $\|x_n\| \to \|x\|$.

5. (30 points) Let $A \in \mathbb{R}^{n\times n}$ be a given matrix.

   (a) Define $\|v\|_1 = \sum_{i=1}^n |v_i|$ for vectors $v \in \mathbb{R}^n$. Show that $\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^n |a_{i,j}|$.

   (b) Define $\|v\|_\infty = \max_{1\le i\le n} |v_i|$ for vectors $v \in \mathbb{R}^n$. Show that $\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^n |a_{i,j}|$.

   (c) Define $\|v\|_2^2 = \sum_{i=1}^n |v_i|^2$. Show that $\|A\|_2 = \max_{1\le i\le n} \sqrt{\lambda_i(A^TA)}$, where $\{\lambda_i(A^TA)\}$ are the eigenvalues of A^TA.

6. (10 points) Let P_n be the vector space of polynomials of degree $k \le n$. For polynomial $p(x) = \sum_{i=0}^n a_i x^i$, define $\|p\| = \max_{0\le i\le n} |a_i|$. Show that this defines a norm on P_n. What is the operator norm of the differentiation operator, D, on this space?