Tufts University
Department of Mathematics
Math 250-03 Homework 5

Due: Thursday, November 1, at 3:00 p.m. (in class).

1. (10 points) Let $A \subset B \subset C$ be normed spaces with common norm $\|\cdot\|$. Prove that if A is dense in B and B is dense in C, then A is dense in C.

2. (20 points) Prove that
$$2\sum_{n=1}^{N} \sin(nx)\sin(nu) = \frac{\sin(N+1/2)(u-x)}{2\sin(u-x)/2} - \frac{\sin(N+1/2)(u+x)}{2\sin(u+x)/2}.$$
Hint: One way to do this is to write $\sin\theta = \frac{e^{\imath\theta}-e^{-\imath\theta}}{2\imath}$.

3. (30 points)
   (a) Apply Gram-Schmidt to $\{x^n\}_{n\geq 0}$ in $L_2[0,1]$, computing the first 3 orthonormal polynomials.
   (b) Find the quadratic polynomial, $Q(x)$, that minimizes $\int_0^1 (\sin(\pi x) - Q(x))^2\, dx$. What is $\|\sin(\pi x) - Q(x)\|^2$?
   (c) Find the quadratic Taylor polynomial, $P(x)$, for $\sin(\pi x)$ around 0. What is $\|\sin(\pi x) - P(x)\|^2$?

4. (10 points) Show that the orthogonal complement of the set of even functions in $L_2[-1,1]$ is the set of odd functions in $L_2[-1,1]$.

5. (10 points) Let $\mathcal{H}$ be a Hilbert space, and $M \subset N \subset \mathcal{H}$. Show that $N^\perp \subset M^\perp$.

6. (20 points) Consider the space $C[-1,1]$ with inner product $\langle u, v\rangle = \int_{-1}^1 u(x)v(x)dx$. Show that the functional $f(u) = \int_0^1 u(x)dx$ is linear and continuous. For what function, $z(x)$, is $f(u) = \langle u, z\rangle$? Why does this not contradict the Riesz Representation Theorem?