Tufts University
Department of Mathematics
Math 250-03 Homework 3

Due: Thursday, October 4, at 3:00 p.m. (in class).

1. (20 points) Find an explicit series solution for

$$\begin{cases} u_t = u_{xx} + e^{-t}\sin(3\pi x) & 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 & t \geq 0 \\ u(x,0) = x\sin(\pi x) & 0 \leq x \leq 1 \end{cases}.$$

2. (20 points) Read Section 5.6.2 in Pinchover and Rubinstein on wine cellars. This is, allegedly, one of the applications that Fourier himself considered.

   Consider the problem of building a reliable wine cellar in Death Valley, CA. For the sake of using round numbers, suppose the average annual temperature is 25°C, with a seasonal fluctuation of ±15°C and a daily fluctuation of ±8°C. Thus, we model the surface temperature after t days as

$$u(0,t) = 25 + 15\cos(\omega_0 t) + 8\cos(\omega_1 t),$$

   where $\omega_0 = (2\pi/365)$ to give one full cycle of the seasonal variation in 365 days, and $\omega_1 = 2\pi$ to give one full cycle of the daily variation each day. We couple this with the heat equation,

$$u_t = \mathcal{K}u_{xx},$$

   for all t, $x > 0$, and the boundedness condition that $u(x,t) \to 25$ as $x \to \infty$ for all t.

   (a) Verify that $u(x,t) = 25 + 15e^{-k_0 x}\cos(\omega_0 t - k_0 x) + 8e^{-k_1 x}\cos(\omega_1 t - k_1 x)$ is a solution to this equation, where $k_0 = \sqrt{\omega_0/(2\mathcal{K})}$ and $k_1 = \sqrt{\omega_1/(2\mathcal{K})}$.

   (b) Assume that the upper layers of the Earth in Death Valley are made of sandstone, with a thermal diffusivity of $\mathcal{K} \approx 1.15 \times 10^{-6} m^2/s \approx 1 m^2/day$. How deep should a cellar be dug so that the daily and annual temperature flucuations are less than one degree?

3. (30 points) Using an energy argument, prove uniqueness of solutions to

$$\begin{cases} u_t = \mathcal{K}u_{xx} + F(x,t) & 0 < x < L, t > 0 \\ u(0,t) + \alpha u_x(0,t) = a(t) & t \geq 0 \\ u(L,t) + \beta u_x(L,t) = b(t) & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases},$$

   where $\alpha < 0 < \beta$. Be sure to indicate your assumptions on the solution.

   Note: in contrast to the examples we considered in class, you won't be able to get rid of the endpoint terms in the integration-by-parts step. You don't need to - just show that $E(t) = \frac{1}{2}\int_0^L w^2 dx$ is still an energy function.

4. (30 points) Using an energy argument, prove uniqueness of solutions to the damped wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} + \alpha u_{txx} & 0 < x < L, t > 0 \\ u(0,t) = a(t) & t \geq 0 \\ u(L,t) = b(t) & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \\ u_t(x,0) = g(x) & 0 \leq x \leq L \end{cases},$$

where $c^2 > 0$ and $\alpha \geq 0$. Be sure to indicate your assumptions on the solution.

Notice that the case $\alpha = 0$ is the same as the undamped equation that we considered in class. This suggests that you might try the same energy function for the general (damped) case.