Tufts University
Department of Mathematics
Math 250-03 Homework 2

Due: Thursday, September 27, at 3:00 p.m. (in class).

1. (20 points) Find an explicit series solution for

$$\begin{cases} u_t = \mathcal{K} u_{xx} & 0 < x < 1, t > 0 \\ u(0,t) = u(1,t) = 0 & t \geq 0 \\ u(x,0) = x(1-x) & 0 \leq x \leq 1 \end{cases}.$$

2. (20 points) Find the general series solution for

$$\begin{cases} u_t = \mathcal{K} u_{xx} & 0 < x < L, t > 0 \\ u(0,t) = u_x(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases},$$

assuming that $f(0) = f'(L) = 0$.

3. (40 points) Let $u(x,t)$ satisfy the heat equation

$$\begin{cases} u_t = \mathcal{K} u_{xx} & 0 < x < L, t > 0 \\ u(0,t) = u(L,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases},$$

where

$$\frac{2}{L} \int_0^L |f(x)|\, dx = N < \infty.$$

Define $B_1 = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi x}{L}\right) dx$. Assume that the formal series solution is well-defined and converges absolutely for all x and t.

Prove each of the following (you may assume all prior steps in proving each):

(a) For any $t > 0$ and $0 < x < L$,

$$\left| u(x,t) - B_1 e^{-\mathcal{K}\left(\frac{\pi}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right) \right| \leq \sum_{n=2}^{\infty} N e^{-\mathcal{K}\left(\frac{n\pi}{L}\right)^2 t}.$$

(b)

$$\sum_{n=2}^{\infty} N e^{-\mathcal{K}\left(\frac{n\pi}{L}\right)^2 t} = N e^{-\mathcal{K}\left(\frac{2\pi}{L}\right)^2 t} \sum_{m=0}^{\infty} e^{-\mathcal{K}\left(\frac{\pi}{L}\right)^2 m(m+4)t}.$$

(c)

$$\sum_{m=0}^{\infty} e^{-\mathcal{K}\left(\frac{\pi}{L}\right)^2 m(m+4)t} \leq \frac{1}{1 - e^{-\mathcal{K}\left(\frac{2\pi}{L}\right)^2 t}}.$$

(d) For every $\varepsilon > 0$, there exists a $\tau > 0$ such that, for all $t > \tau$,

$$\frac{e^{-2\mathcal{K}\left(\frac{\pi}{L}\right)^2 t}}{1 - e^{-\mathcal{K}\left(\frac{2\pi}{L}\right)^2 t}} < \varepsilon.$$

Find such a τ as a function of $\mathcal{K}$, L, and ε.

(e) For every $\varepsilon > 0$, there exists a $\tau > 0$ such that, for all $t > \tau$,

$$\left| u(x,t) - B_1 e^{-\mathcal{K}\left(\frac{\pi}{L}\right)^2 t} \sin\left(\frac{\pi x}{L}\right) \right| \leq \varepsilon e^{-2\mathcal{K}\left(\frac{\pi}{L}\right)^2 t}.$$

Find such a τ as a function of $\mathcal{K}$, L, and ε.

4. (20 points) Consider the heat equation with non-homogeneous boundary conditions:

$$\begin{cases} u_t = \mathcal{K} u_{xx} & 0 < x < L, t > 0 \\ u(0,t) = g_1(t) & t \geq 0 \\ u(L,t) = g_2(t) & t \geq 0 \\ u(x,0) = f(x) & 0 \leq x \leq L \end{cases},$$

and assume that $f(0) = g_1(0) = g_2(0)$.

We can convert this equation into a non-homgeneous equation with homogeneous boundary conditions by writing

$$u(x,t) = w(x,t) + (1 - x/L)g_1(t) + (x/L)g_2(t).$$

Find the PDE satisfied by $w(x,t)$.