MATH 2260 (Ordinary Differential Equations I) — Winter 2015
Homework #2

Due Date: Wednesday, January 28, in class or in marking box #31 by 5:00 PM. You must show all work to receive credit.

1. (10 points) Consider a substance whose mass decays proportionally to the square of the mass present.

   (a) Write an equation to model the mass as a function of time. Use $k > 0$ as the decay constant.

   (b) If 50 grams of the substance are present at time $t = 0$ and 25 grams are present at time $t = 1$, what is the value of k?

   (c) At what time will 10 grams be left?

2. (10 points) A 200 L tank initially contains 100 L of water with 10 kg of salt dissolved in it. From time $t = 0$, a salt solution with 0.1 kg of salt per litre is added at a rate of 4 L per minute, while the resulting mixture is drained at a rate of 2 L per minute. How much salt is in the tank at the moment when it starts to overflow?

3. (10 points) A cup of coffee, initially at 90 C, is left outside on a morning when the temperature is -20 C. After one minute, the temperature of the coffee drops 20 C. How long does it take until the coffee freezes?

4. (10 points) Consider the equation $(x + 1)(9 - x^2)y' + 2xy = \tan(x)$. For each of the following initial conditions, determine (without solving the equation) the interval on which a unique solution exists.

   (a) $y(0) = 0$

   (b) $y(1) = 7$

   (c) $y(2) = 4$

5. (10 points) For each of the following equations, determine the values of (x_0, y_0) where the initial value problem with $y(x_0) = y_0$ has (i) a solution, and (ii) a unique solution on some interval that contains x_0.

   (a) $y' = \frac{x^2+y^2}{\ln xy}$

   (b) $y' = (x^2 + y^4)y^{1/3}$

6. (10 points) Consider the equation $y' = 2xy^2$.

   (a) For what initial values $y(x_0) = y_0$, does the equation have a unique solution on some interval containing x_0?

   (b) Solve the equation with initial value $y(0) = y_0$. What is the interval on which the resulting solution is well-defined? (Note that your answer should depend on y_0.)

7. (40 points) Solve the following ODEs.

   (a) $(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$.

   (b) $y' - 2y = xy^3$, $y(0) = 2\sqrt{2}$.

   (c) $y' = \frac{x^3+y^3}{xy^2}$, $y(1) = 3$.

   (d) $y' = y^2e^{-x} + 4y + 2e^x$. Hint: first factor e^x out of the right-hand side.