Singular perturbation problems in microscopic elastic-electrostatic interfaces

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Numerical analysis of Singularly Perturbed Problems
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Collaborators:

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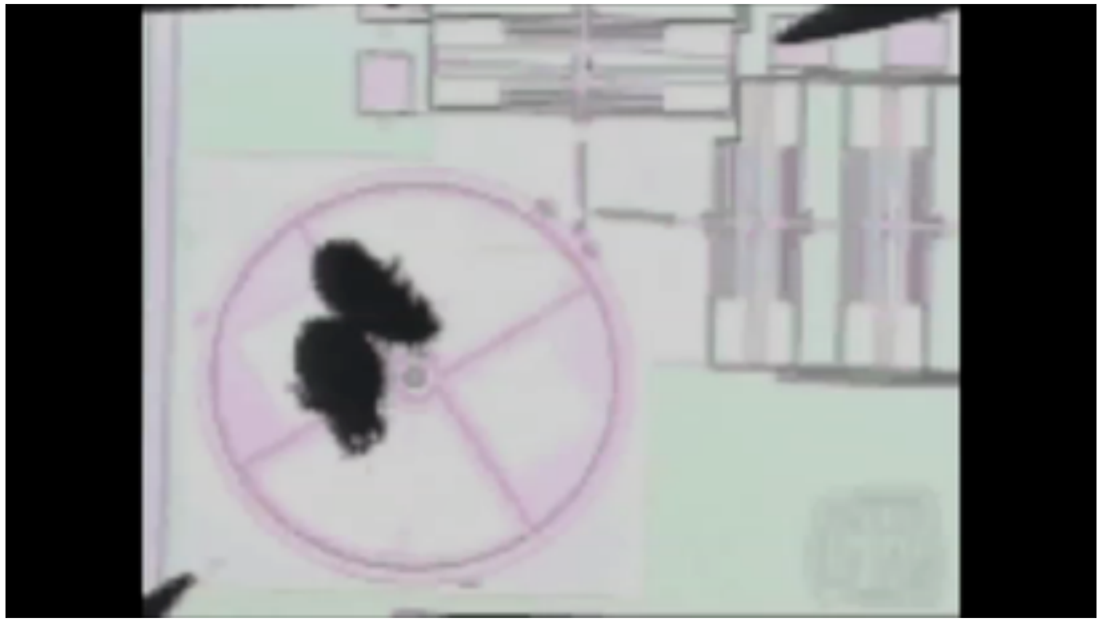
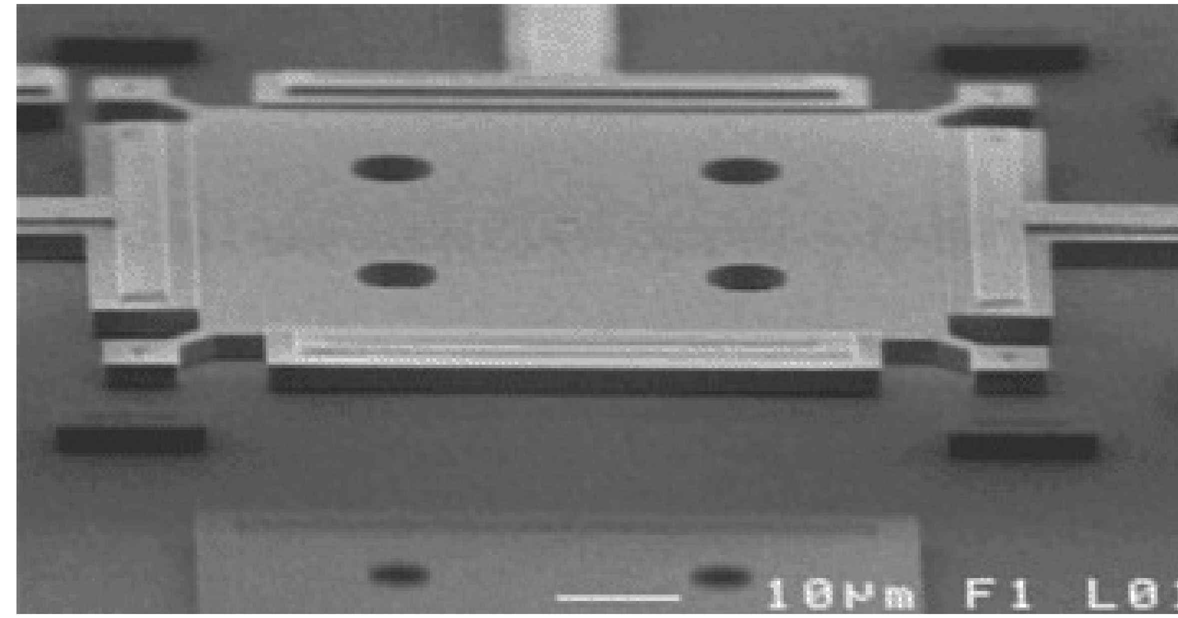
Support:

NSF DMS - 1516753
NSF DGE - 1313583

Micro Electro Mechanical systems (MEMS)

MEMS = (Moving elastic components + Circuitry) x $\mathcal{E}$





Gear and resonators. Source: mems.sandia.gov

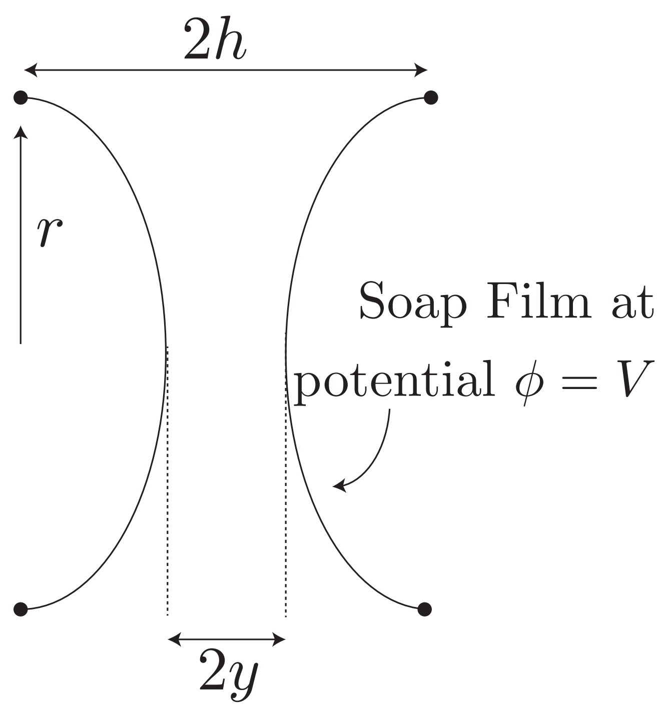
Modeling microscopic dynamical processes:

- Continuum modeling still valid.
- Inertial effects negligible - viscous damping dominates.
- Combustion not practical for locomotion - use electrostatic actuation

Goal: Understand the fundamental process of how elastic surfaces deform in electric fields.

G. I. Taylor (1968) - The Pull-in Instability

- Experimental and theoretical study of liquid drops at different potentials.
  Ref: Proc. Roy. Soc A (1968).



Theoretical Framework

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} = \alpha + \frac{\lambda}{y^2} \qquad \lambda \propto V^2$$

$$y = 1 \quad \text{at} \quad r = 1$$

Experimental Apparatus

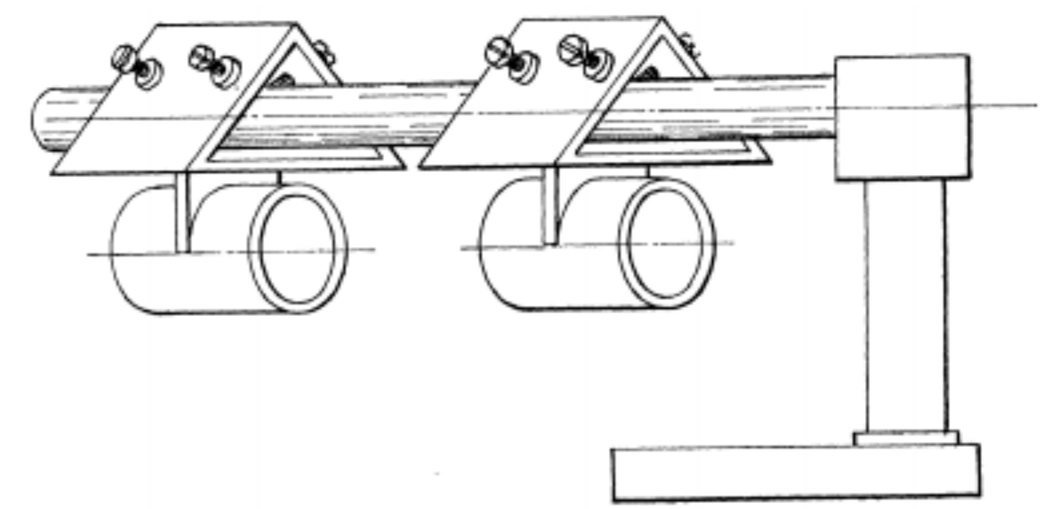
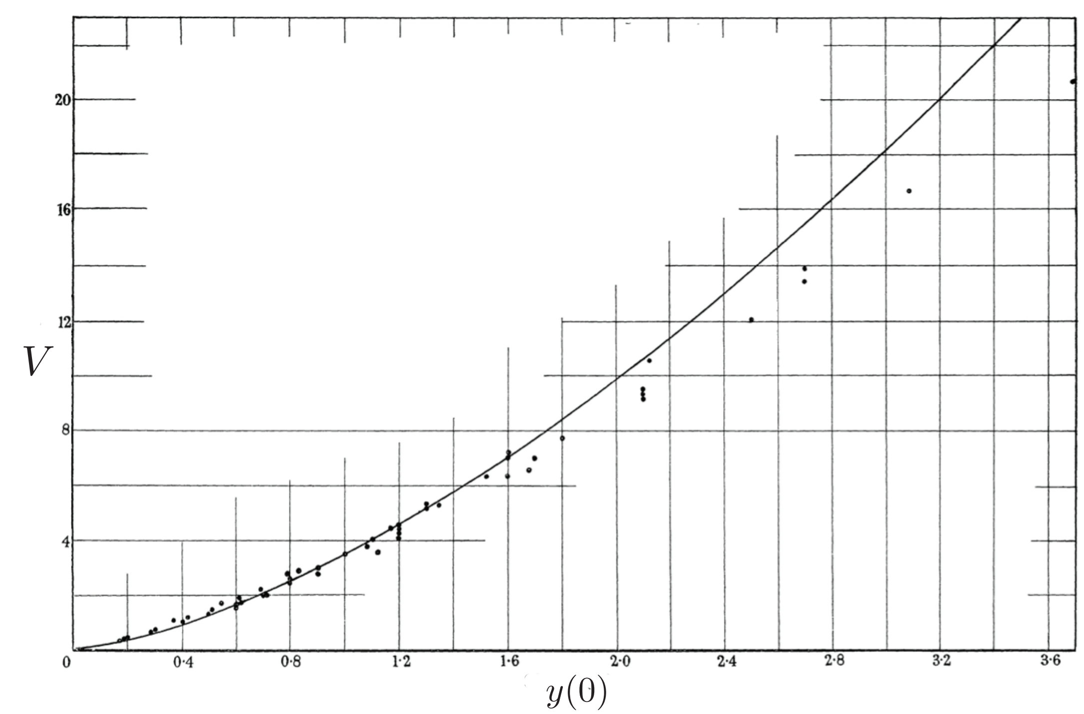
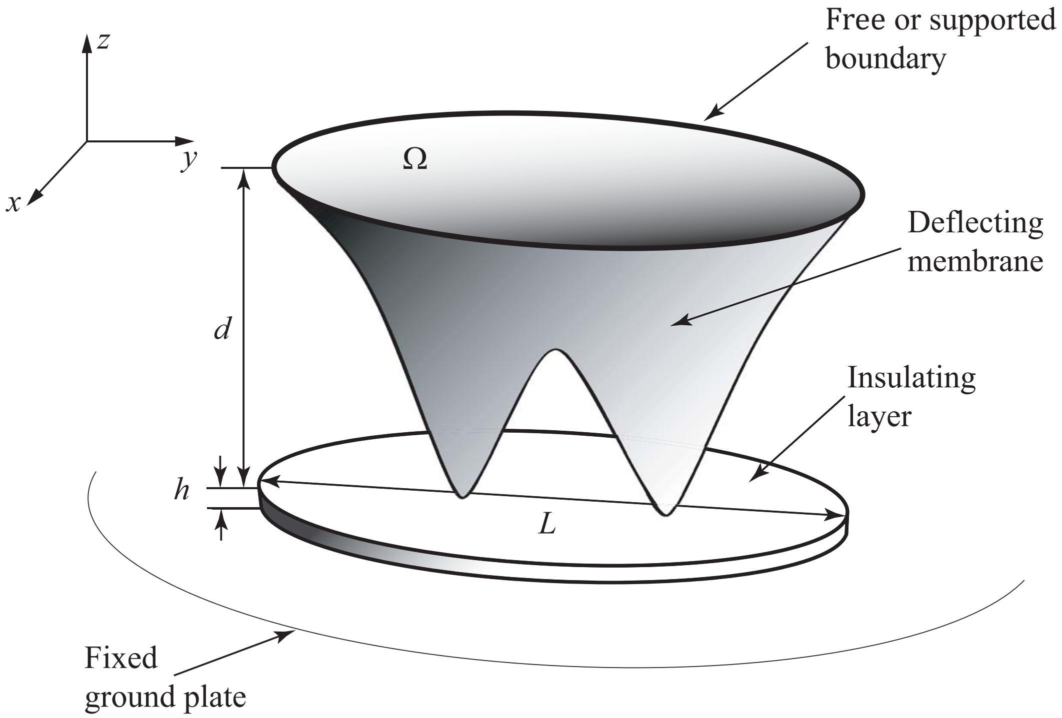


FIGURE 2. Apparatus for holding two circular soap films in position.



Pull-In Instability: elastic surfaces come into physical contact when electric field is large enough to overcome membrane tension.

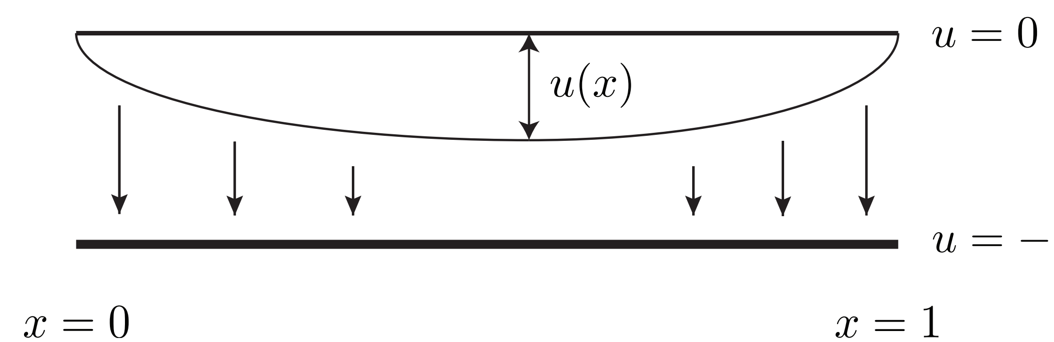
The canonical MEMS problem



- Elastic surface occupying deflecting in the presence of an electric field
- Surfaces come into contact if voltage large enough:
- Small aspect ratio - d/L. Roughly 0.01 in typical MEMS.

Mathematical Model: Pelesko (2003)

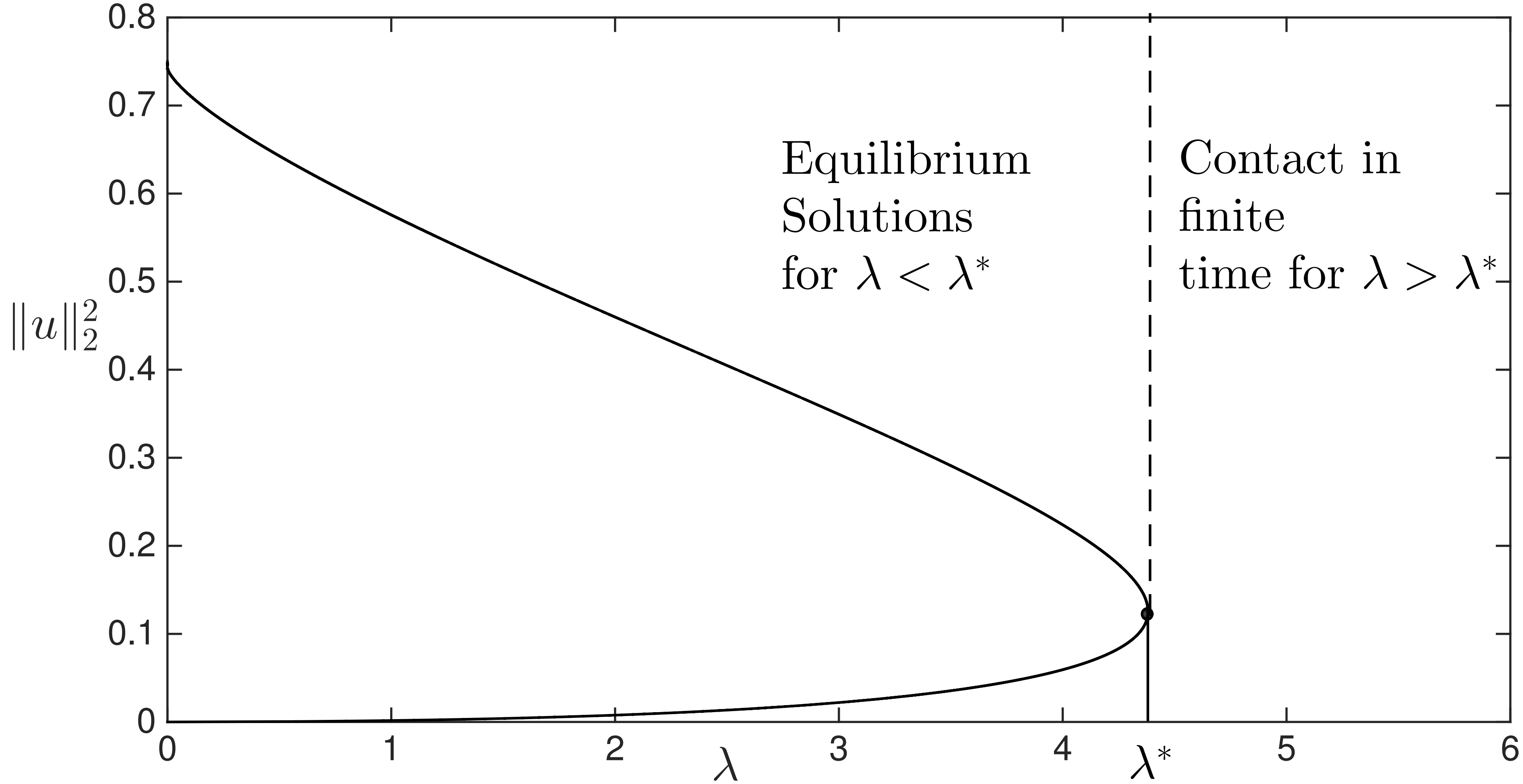
$$\frac{\partial u}{\partial t} = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}, \qquad u|_{\partial\Omega} = 0, \quad \delta \partial_n u|_{\partial\Omega} = 0$$



- Boundary conditions imply zero deflection and clamped at end points.
- $\delta > 0$ - surface is a beam - rigid material.
- $\delta = 0$ - surface is a membrane (soap film).
- $\lambda \propto V^2$ - the main control parameter.
- Contact or touchdown when $u \to -1$

The generic bifurcation diagram

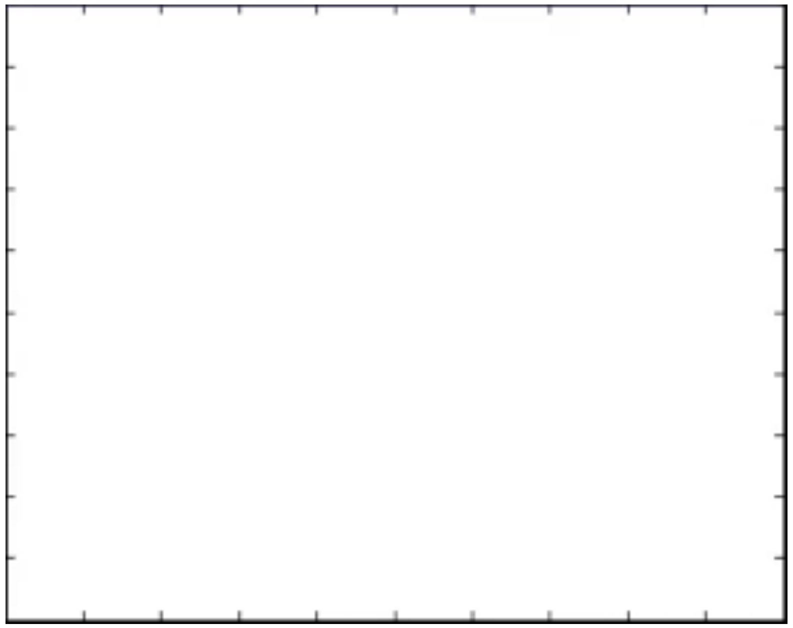
$$\frac{\partial u}{\partial t} = -\delta \Delta^2 u + \Delta u - \frac{\lambda}{(1+u)^2}, \qquad u|_{\partial\Omega} = 0, \quad \delta \partial_n u|_{\partial\Omega} = 0$$



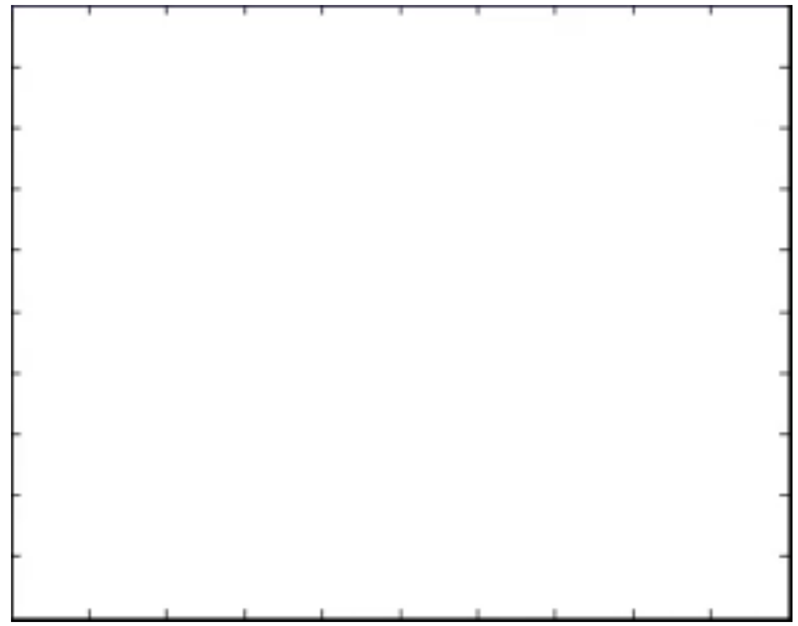
<u>Refs:</u> Brubaker, Cowan, Davila, Escher, Esposito, Flores, Guo, Ghoussoub, Glasner, Hu, Kavallaris, Kohlmann, Lacey, Laurencot, Lega, Lienstromberg, Lindsay, Moradifam, Nikolopoulos, Pan, Pelesko, Ward, Walker, Wei.

Questions: <u>Why</u>, <u>How</u>, <u>When</u>, <u>Where</u> do singularities form and <u>what then</u>?

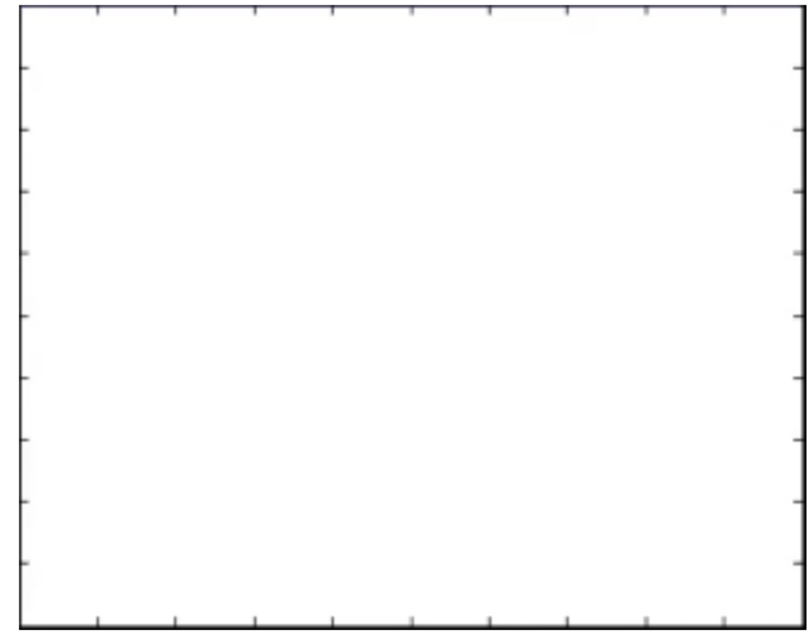
$$u_t = -\Delta^2 u - \frac{\lambda}{(1+u)^2}, \qquad u = \partial_n u = 0, \qquad u(x,0) = 0$$



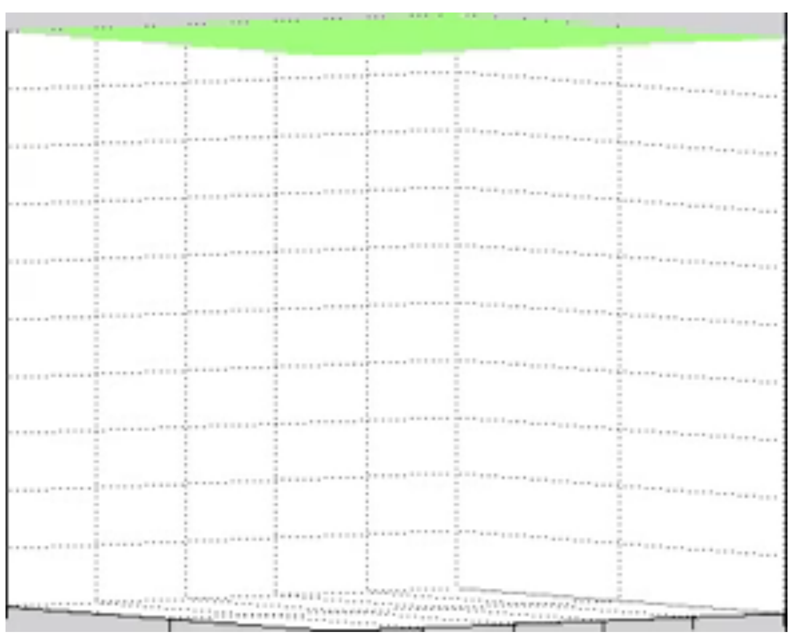
$\lambda = 5$



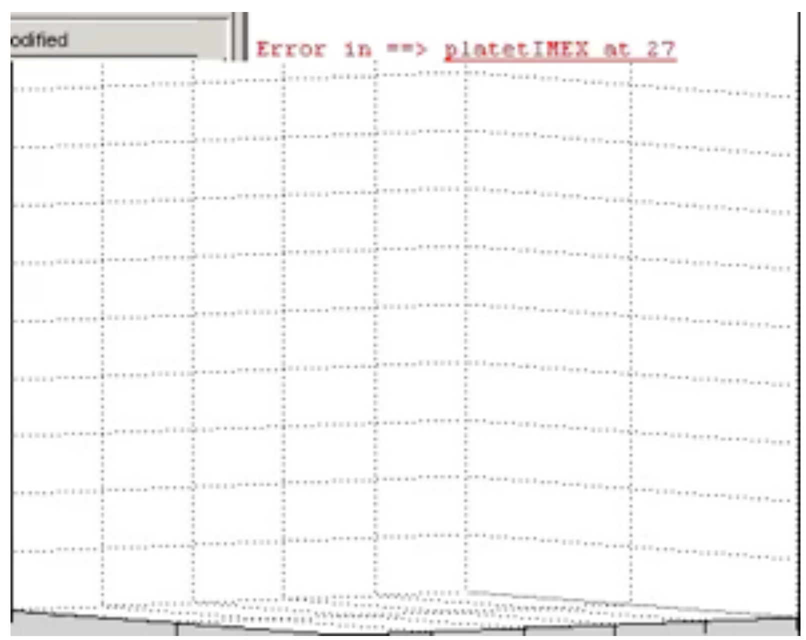
$\lambda = 10$



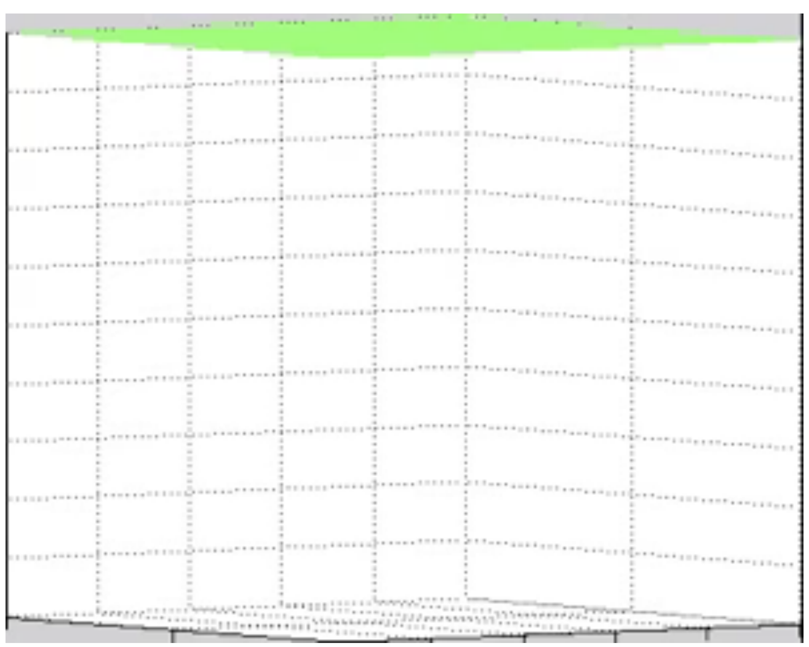
$\lambda = 45$



$\lambda = 3$



$\lambda = 10$



$\lambda = 50$

Outline of Talk

1. Adaptive numerical methods.

    - r-adaptive meshes for generating meshes.
    - Meshes inherit the symmetries/scaling properties of the PDE.

2. Predicting the set of contacts.

    - Concept set complexity described by a boundary layer analysis.
    - Prediction of contact sets in 1D and general 2D regions.

3. Regularized problem describing post contact dynamics.

    - How do we make sense of solutions beyond initial singularities?
    - Layer dynamics and numerical simulations of sharp interfaces.

Adaptive Numerical Methods - Time Adaptation

Motivation: Need to reduce timestep as singularity approached and prevent overshooting the blow up time.

- Scale Invariance

$$t \to a\tau, \qquad x \to a^{\gamma}x, \qquad (1+u) \to a^{\beta}(1+u).$$

- Plug the scaled variables into the MEMS equation:

$$a^{\beta-1}\frac{\partial u}{\partial \tau} = -a^{\beta-4\gamma}\Delta^2 u - a^{-2\beta}\frac{\lambda}{(1+u)^2}, \quad \Longrightarrow \quad \beta = \frac{1}{3}, \quad \gamma = \frac{1}{4}$$

- This gives the scaling law:
  $(1+u(x,t)) = a^{\frac{1}{3}}\left(1+u\left(\frac{x}{a^{1/4}}, \frac{t}{a}\right)\right)$
- Choosing the adaptive time step: $\boxed{dt = d\tau \min(1+u)^3}$

Spatial Adaptivity - Three classes of adaptive methods.

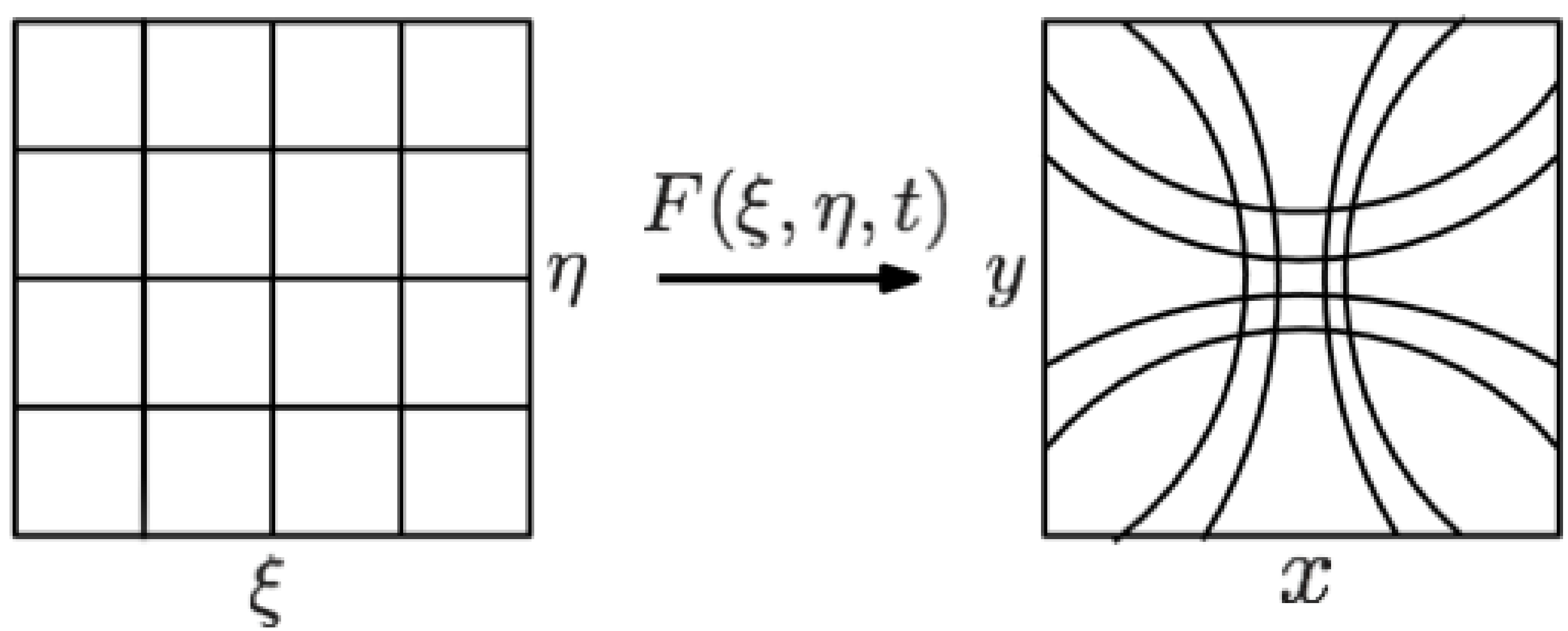
- h-adaptive - Add mesh points to regions where extra resolution required (singularities). Remove mesh points where less solution resolution is needed.

- p-adaptive - Increase the order of the approximating functions

- r-adaptive - Move fixed number of mesh points to spatial regions where more accuracy is required. [Budd2006,Hou2001, BuddJFW2009,HuangRussel2011].

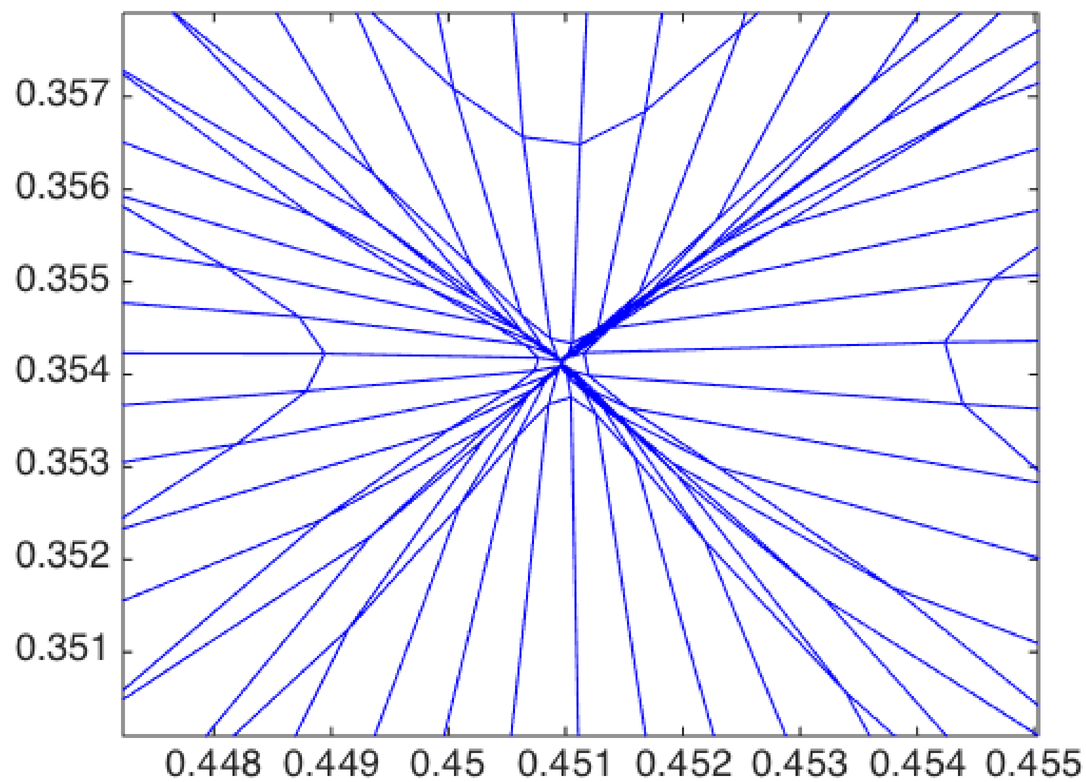
R-Adaptive Methods

- Two components to moving mesh derivation:
  1. Describing the optimal mesh
  2. Develop a strategy for evolving the mesh to the optimal mesh
- Steps for an R-adaptive mesh:
  - Start with a fixed number of mesh points.
  - Find a continuous mapping, $X = F(\xi, t)$, between the computational space and physical space; i.e. $\Omega_C \to \Omega_P$



Mesh Tangling

- Need $F(\boldsymbol{\xi}, t)$ to be a 1-1 mapping to avoid mesh tangling
- 1-1 mapping implies: $|J(\boldsymbol{\xi}, t)| = \det\left(\frac{\partial \mathbf{X}(\boldsymbol{\xi}, t)}{d\boldsymbol{\xi}}\right) = \begin{vmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{vmatrix} > 0$



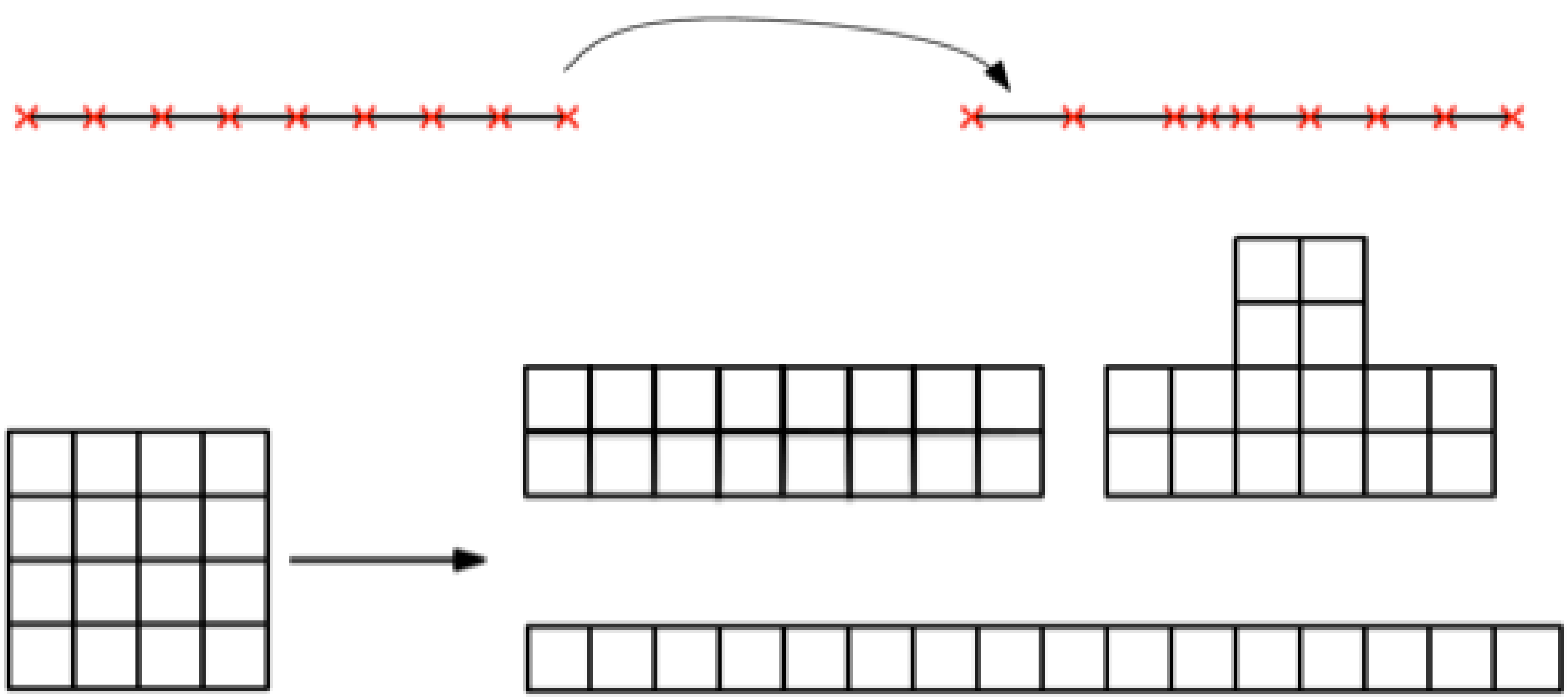
An example of mesh tangling.

Finding an Optimal Mesh

- For any invertible F, we can find a *Monitor Function* $M(\mathbf{x})$ that for any $A \subset \Omega_C$

$$\int_A d\mathbf{x} = \int_{F(A)} M(\mathbf{x}) d\mathbf{x}$$

- The idea is to equidistribute $M(\mathbf{x})$ over the mesh.
- In 1D equidistribution defines a unique mesh, but <u>not in 2D</u>.



Finding the Mapping F in 2D

- Want want to choose the mapping that is closest to a uniform mesh by minimizing the least squares norm

$$I = \int_{\Omega_C} |F(\boldsymbol{\xi}, t) - \boldsymbol{\xi}|^2 d\boldsymbol{\xi}$$

Theorem (Delzanno 2008)

There exists a unique optimal mapping $\mathbf{F}(\xi, t)$, satisfying the equidistribution equation. The map has the same regularity as M. Furthermore, $\mathbf{F}(\xi, t)$ is the unique mapping from this class which can be written as the gradient (with respect to ξ) of a convex (mesh) potential $P(\xi, t)$, so that:

$$\mathbf{F}(\xi, t) = \nabla_\xi P(\boldsymbol{\xi}, t), \quad \Delta_\xi P(\xi, t) > 0$$

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The Nonlinear Monge-Ampere Equation

- Assuming $\mathbf{X} = \nabla_\xi P$, then $\frac{\partial \mathbf{X}}{\partial \boldsymbol{\xi}} = |H(P)| = P_{\xi\xi}P_{\eta\eta} - P_{\xi\eta}^2$
- The equidistribution principle gives the Nonlinear Monge-Ampere Eqn.

$$M(\nabla_\xi P, t)|H(P)| = \frac{\int_{\Omega_P} M(\mathbf{X}(\boldsymbol{\xi}, t) d\mathbf{X}}{\int_{\Omega_C} d\boldsymbol{\xi}}$$

- Problem: Monge-Ampere equation is a fully nonlinear PDE.
- Solution: Can solve approximately using a parabolic equation for $Q(\xi, t)$ which evolves toward the gradient of $P(\xi, t)$.
- Relaxed equation (PMA equation):

$$\alpha(I - \gamma\Delta)Q_t = (|H(Q)|M(\nabla Q))^{1/d}$$

- $\alpha = 0.1$ - speed of relaxation. $\gamma = 0.1$ - smoothing parameter.

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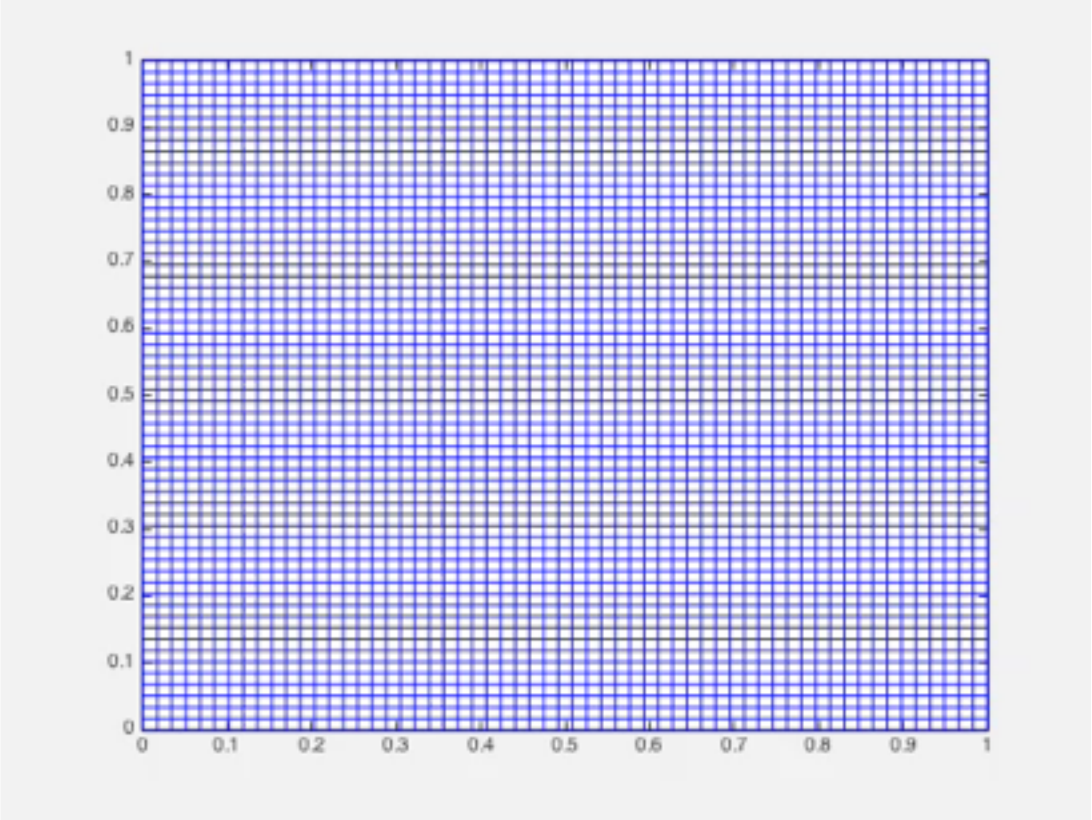
Choosing the Monitor Function

- *A priori* estimates based on physics or geometry.
- ex: Arclength $\sqrt{1 + c^2 |\nabla_{\boldsymbol{\xi}} u(x(\xi))|^2}$
- *A posteriori* estimates based on error can also be used.
- For strong scaling structures, want a monitor function that scales with the problem
- For MEMS Problem: $M(u) = \dfrac{1}{(1+u)^3}$ in 1D and $M(u) = \dfrac{1}{(1+u)^6}$ in 2D.

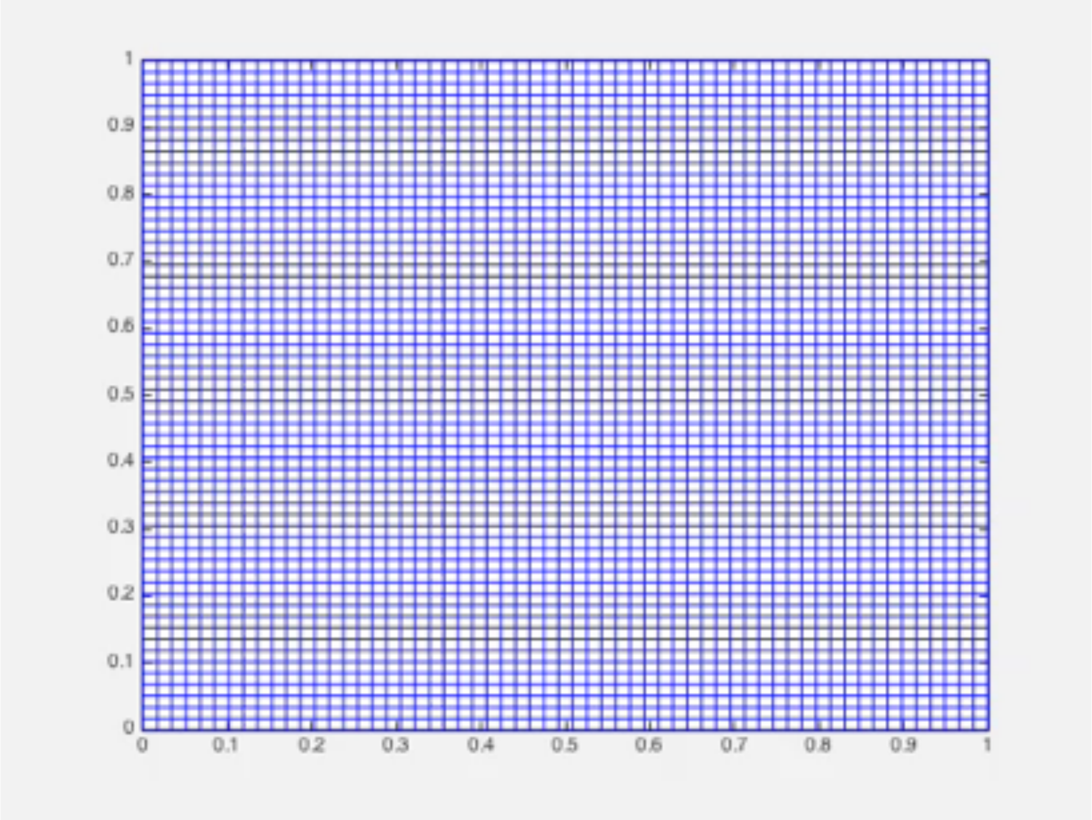
Efficient Discretization: Regularization and Smoothing.

- Problem: Rushing of a majority of Mesh Points to singularities.
- Use a McKenzie Regularization to distribute half the mesh points around singularities and remainder elsewhere.

$$\bar{M} = M + \int_\Omega M(\mathbf{X}, t) d\mathbf{X}$$



No regularization

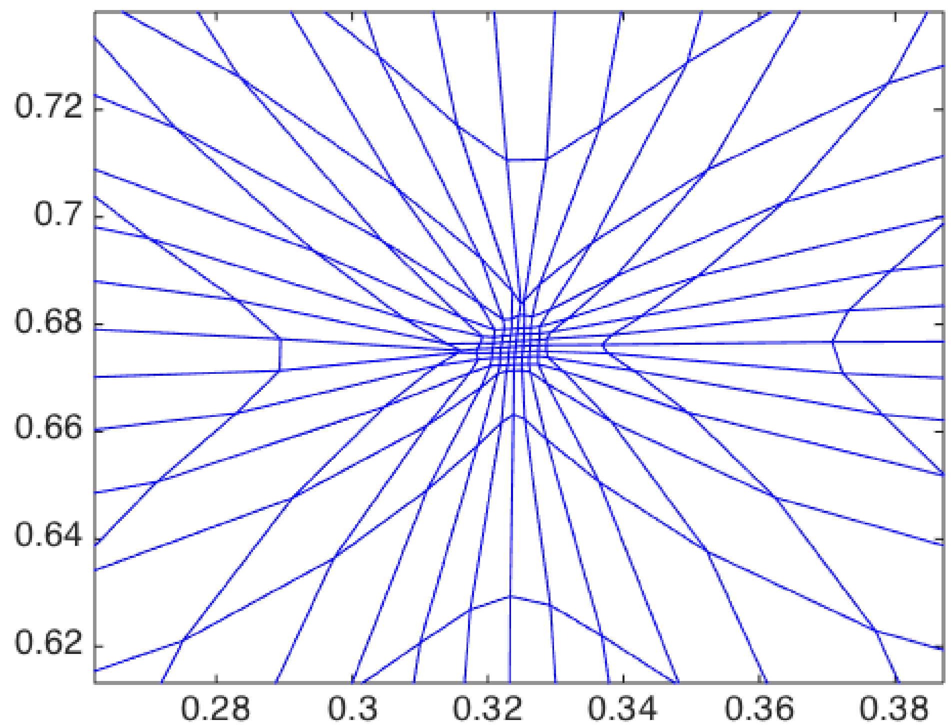


Regularized

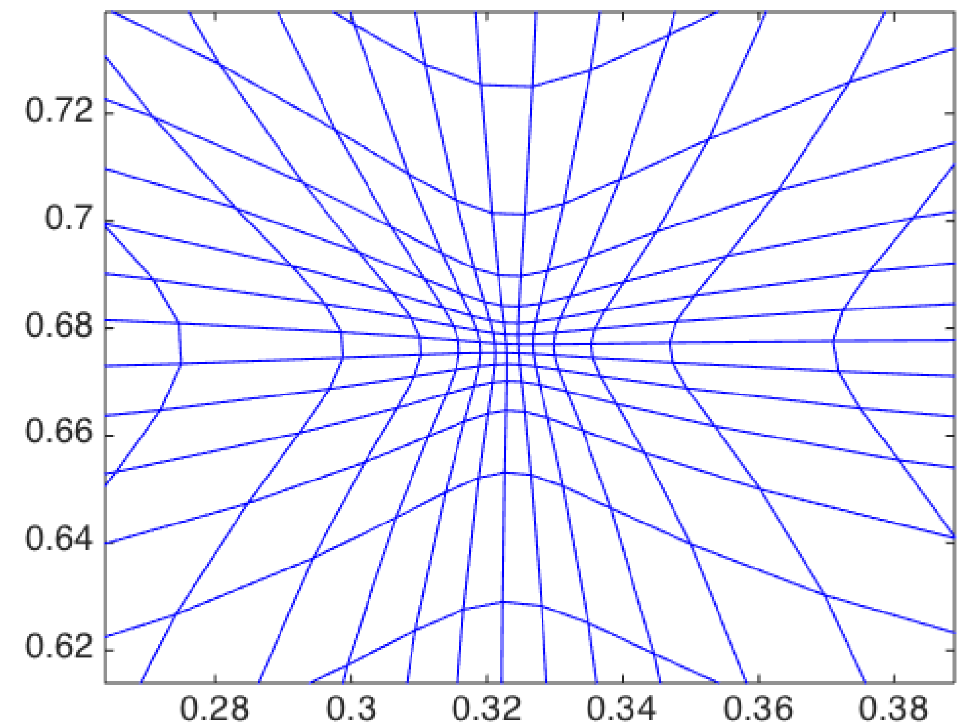
Smoothing of Monitor Function

- Problem: Smoothness required for reliable differentiation.
- Solution: Apply a fourth order smoothing filter.

$$M_{i,j} \leftarrow \frac{4}{16} M_{i,j} + \frac{2}{16}(M_{i+1,j} + M_{i-1,j} + M_{i,j-1} + M_{i,j+1})$$
$$+ \frac{1}{16}(M_{i+1,j+1} + M_{i-1,j-1} + M_{i+1,j-1} + M_{i-1,j+1})$$



(i) No Smoothing

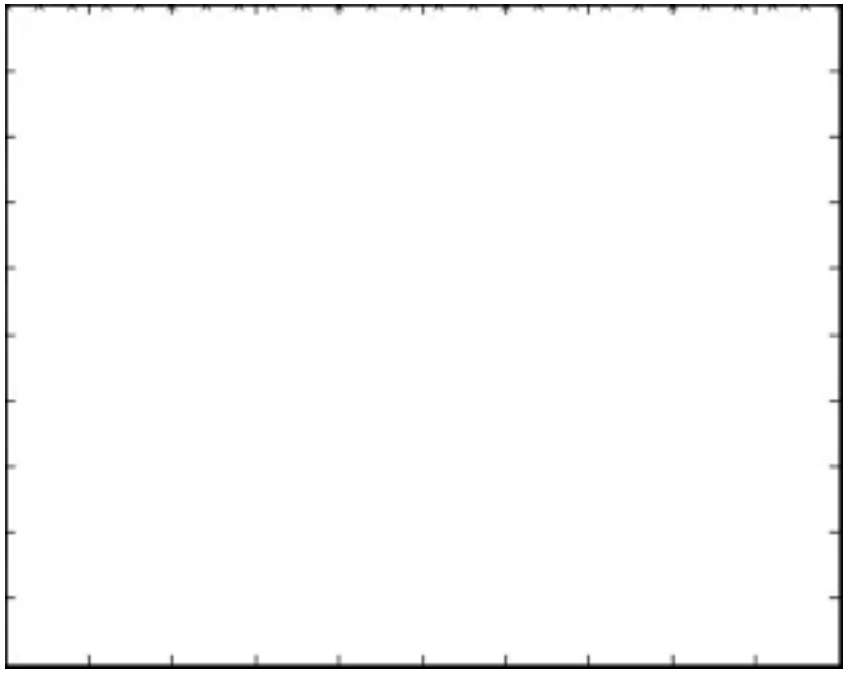


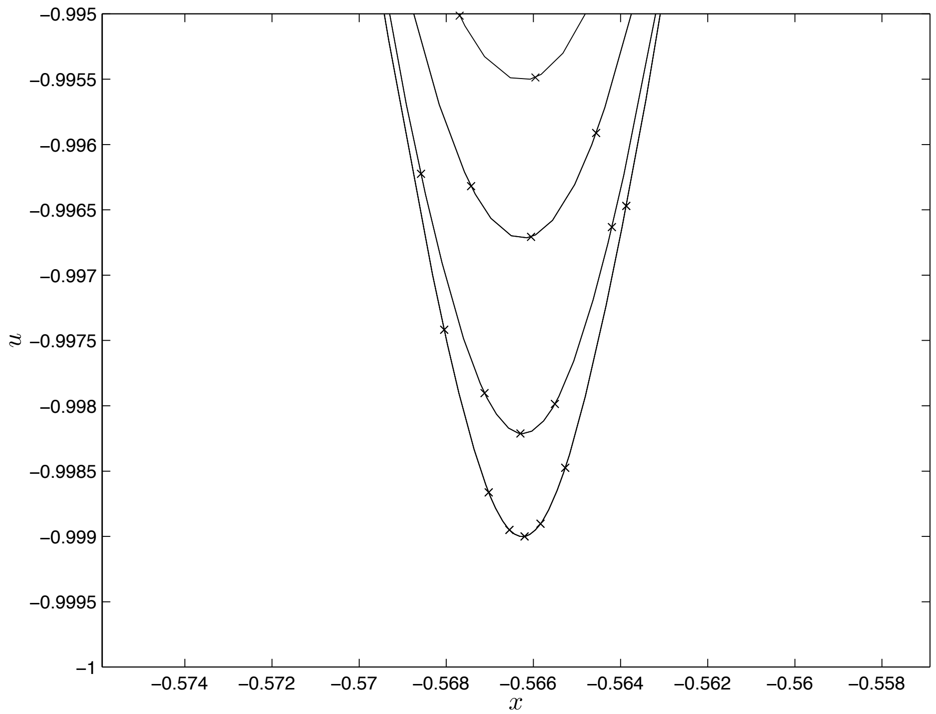
(j) Smoothing

One-dimensional simulations: Using MOVCOL4 (Russel,Xu,Williams)

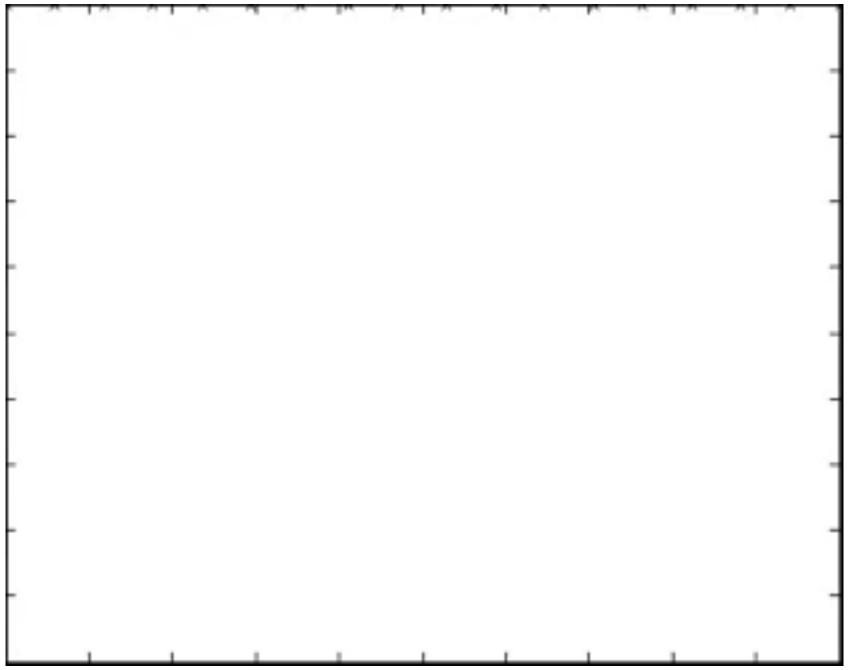
$$u_t = -\Delta^2 u - \frac{\lambda}{(1+u)^2}$$

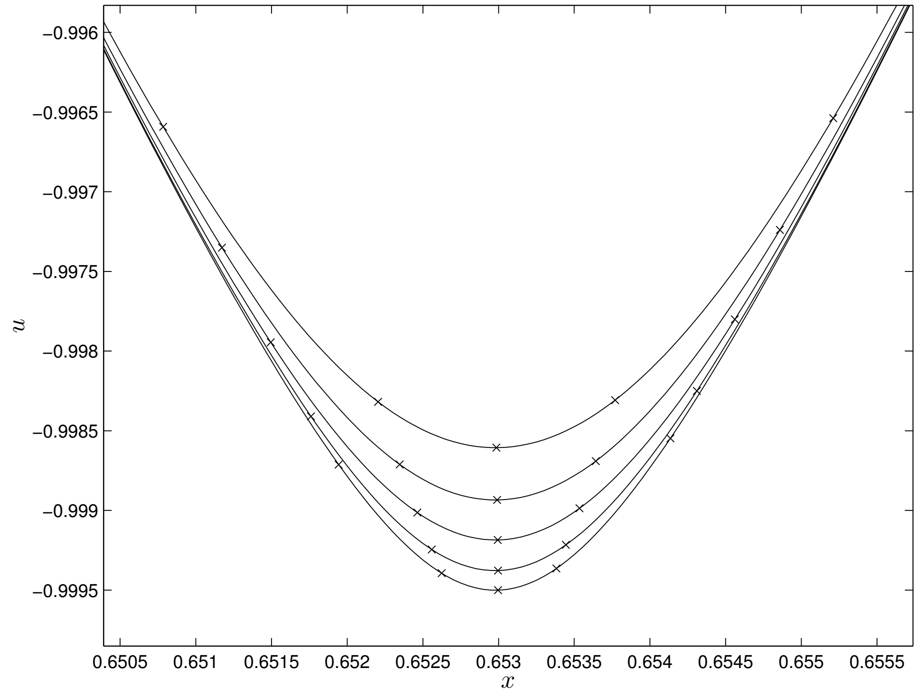
1D:



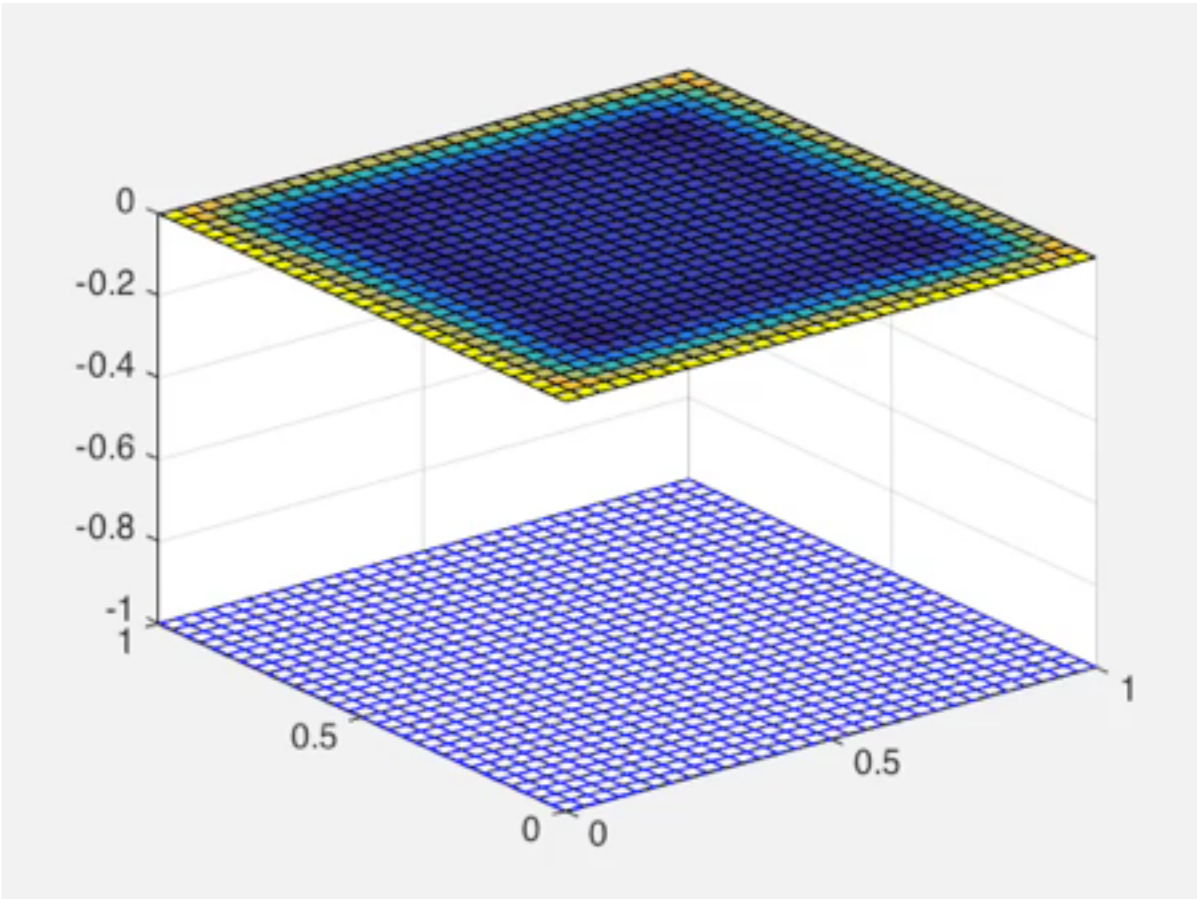


Unit Disk

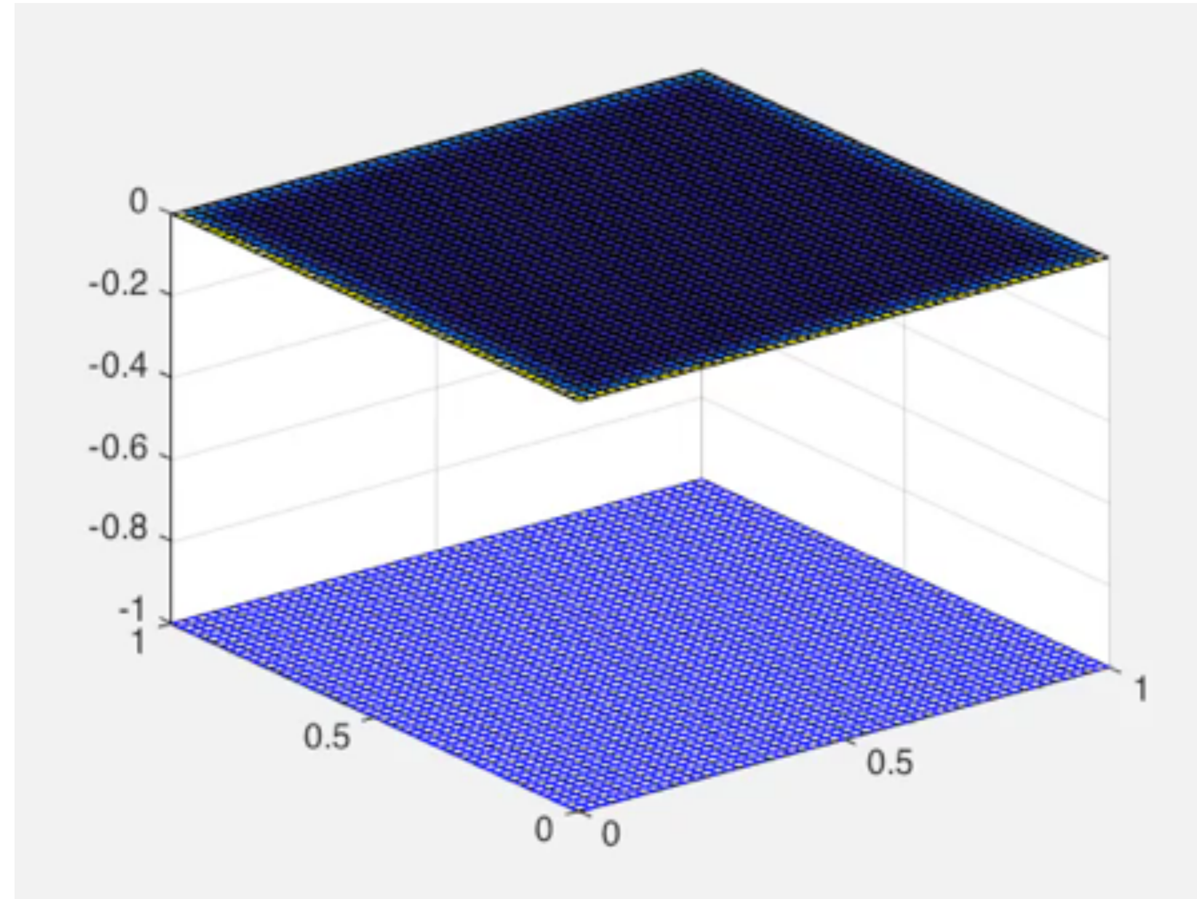




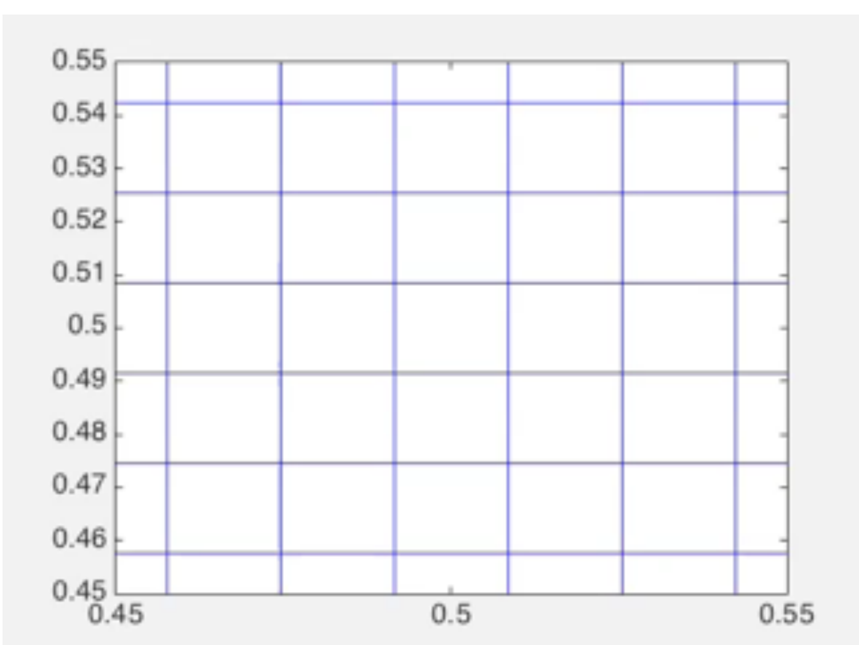
2D Results Using Monge Ampere Adaptation



$\lambda = 15$



$\lambda = 45$



$$u_t = -\Delta^2 u - \frac{\lambda}{(1+u)^2}$$

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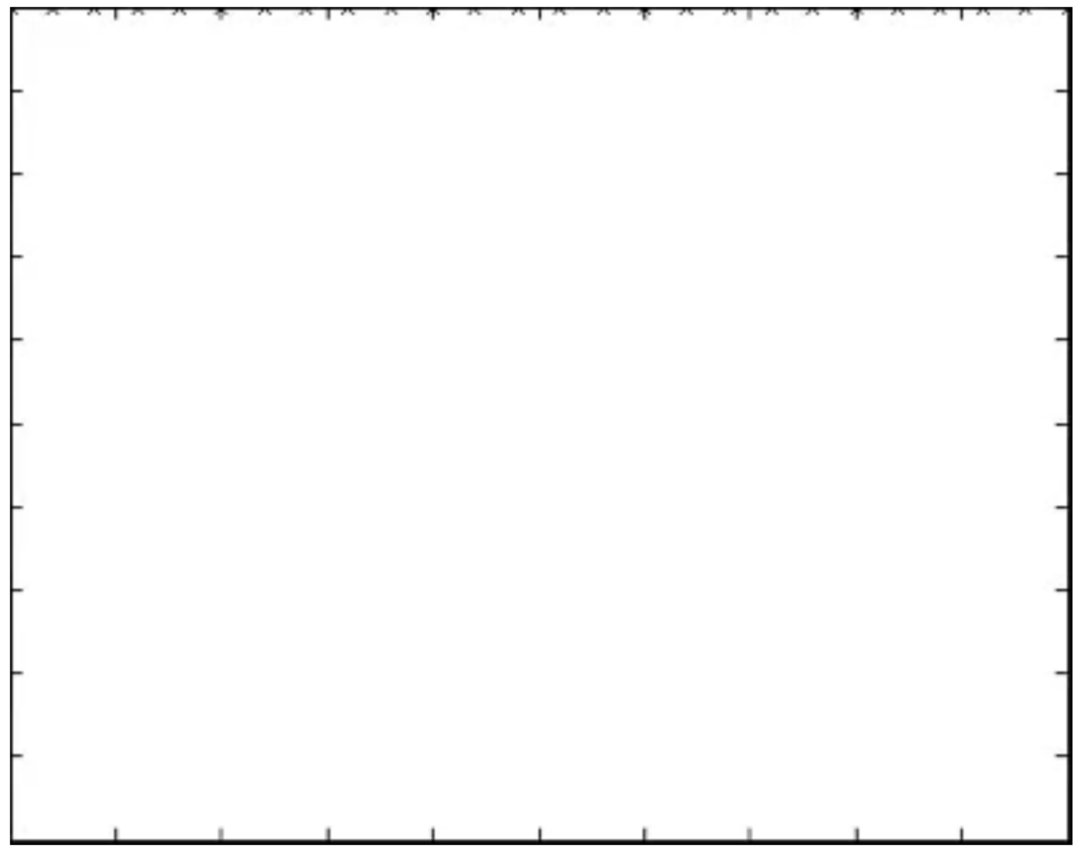
Rescaling to obtain a singular perturbation problem.



- Rescale with $t \to \lambda^{-1} t$, $\quad \varepsilon^2 = \lambda$, $\quad f(u) = -\dfrac{1}{(1+u)^2}$

$$\boxed{u_t = -\varepsilon^2 \Delta^2 u + f(u), \quad x \in \Omega; \qquad u = \partial_n u = 0, \quad x \in \partial\Omega}$$

Basis of Analysis

Small t behaviour:

- Flat central region coupled to a propagating boundary effect.
- In the flat central region, $u(x,t) \sim f(t)$;

$$f_t = -\frac{1}{(1+f)^2}, \qquad f = -1 + (1-3t)^{1/3}$$

- Propagating boundary effect (at $x = 1$) in stretching coordinates:

$$u(x,t) \sim -f(t)\, v_0(\eta) \qquad \eta = \frac{1-x}{\varepsilon^{1/2} f^{1/4}}$$

- $t \to 0$ corresponds to $f \to 0$ so the $(1+u)^{-2}$ term is linearized.

Touchdown Behaviour: Small $(t_c - t)$

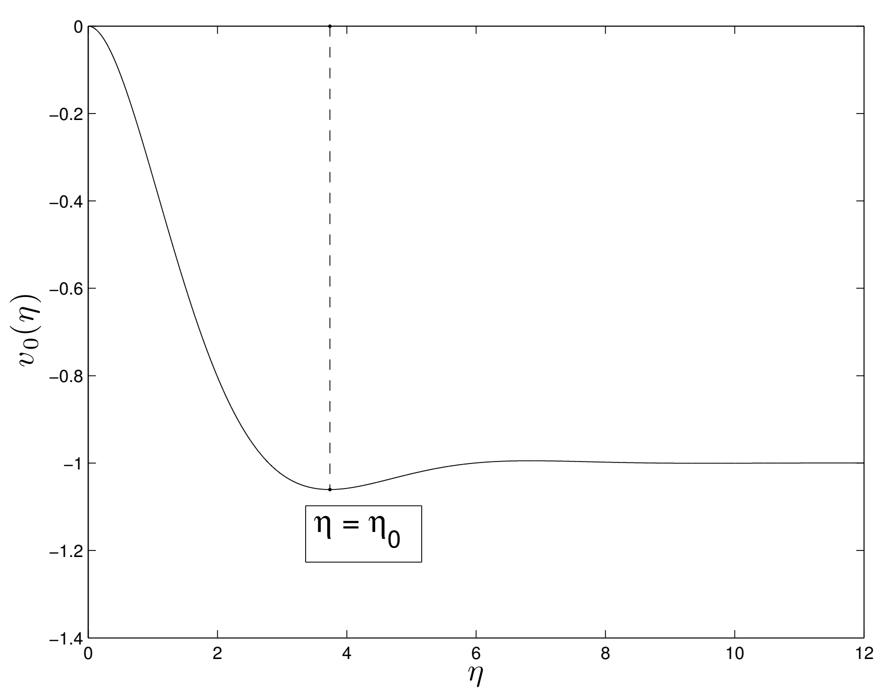
- $t_c(\varepsilon)$ is the finite touchdown time.

Stretching Boundary Layer:

After analysis:

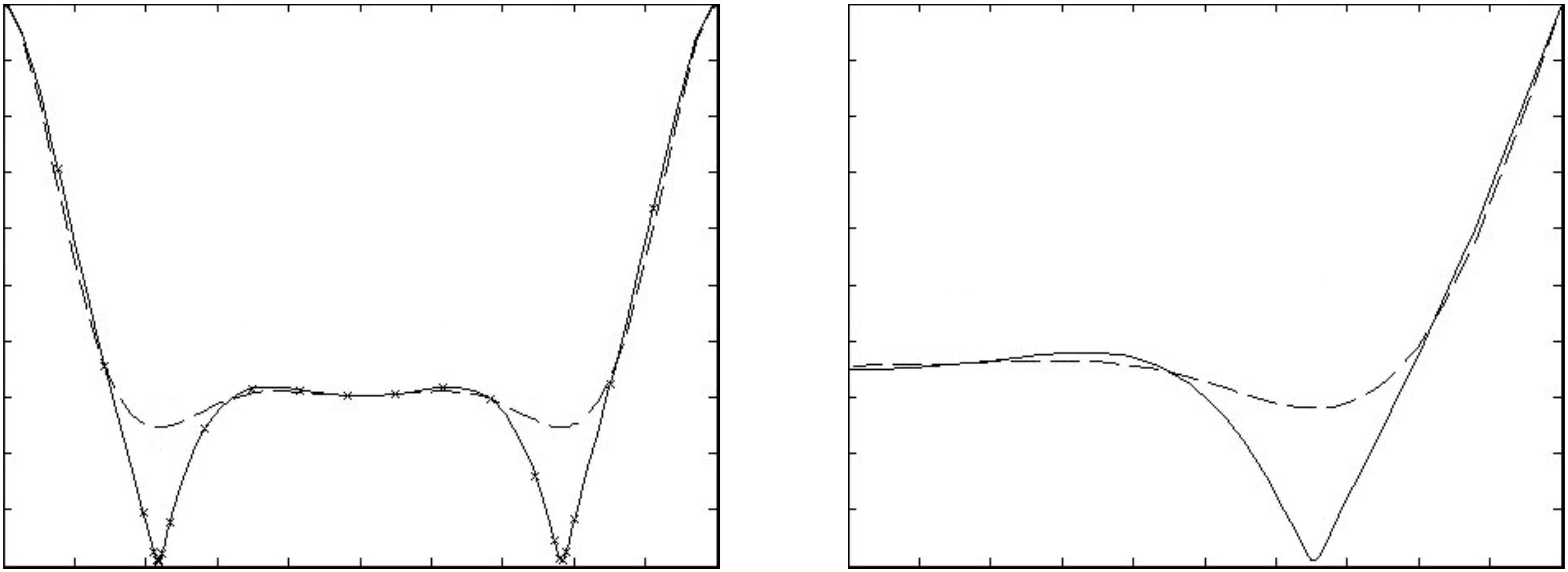
$$\frac{d^4 v_0}{d\eta^4} - \frac{\eta}{4}\frac{dv_0}{d\eta} + v_0 = -1, \quad \eta > 0; \qquad v_0 \sim -1 \quad \text{as} \quad \eta \to \infty$$

Solution:



$$u(x,t) \sim \underbrace{f(t)}_{Outer} - \underbrace{f(t)\left[v_0\left(\frac{1-x}{\varepsilon^{1/2}f^{1/4}}\right) + v_0\left(\frac{x-1}{\varepsilon^{1/2}f^{1/4}}\right)\right]}_{Boundary\,Terms} - 2\underbrace{f(t)}_{Overlap}$$

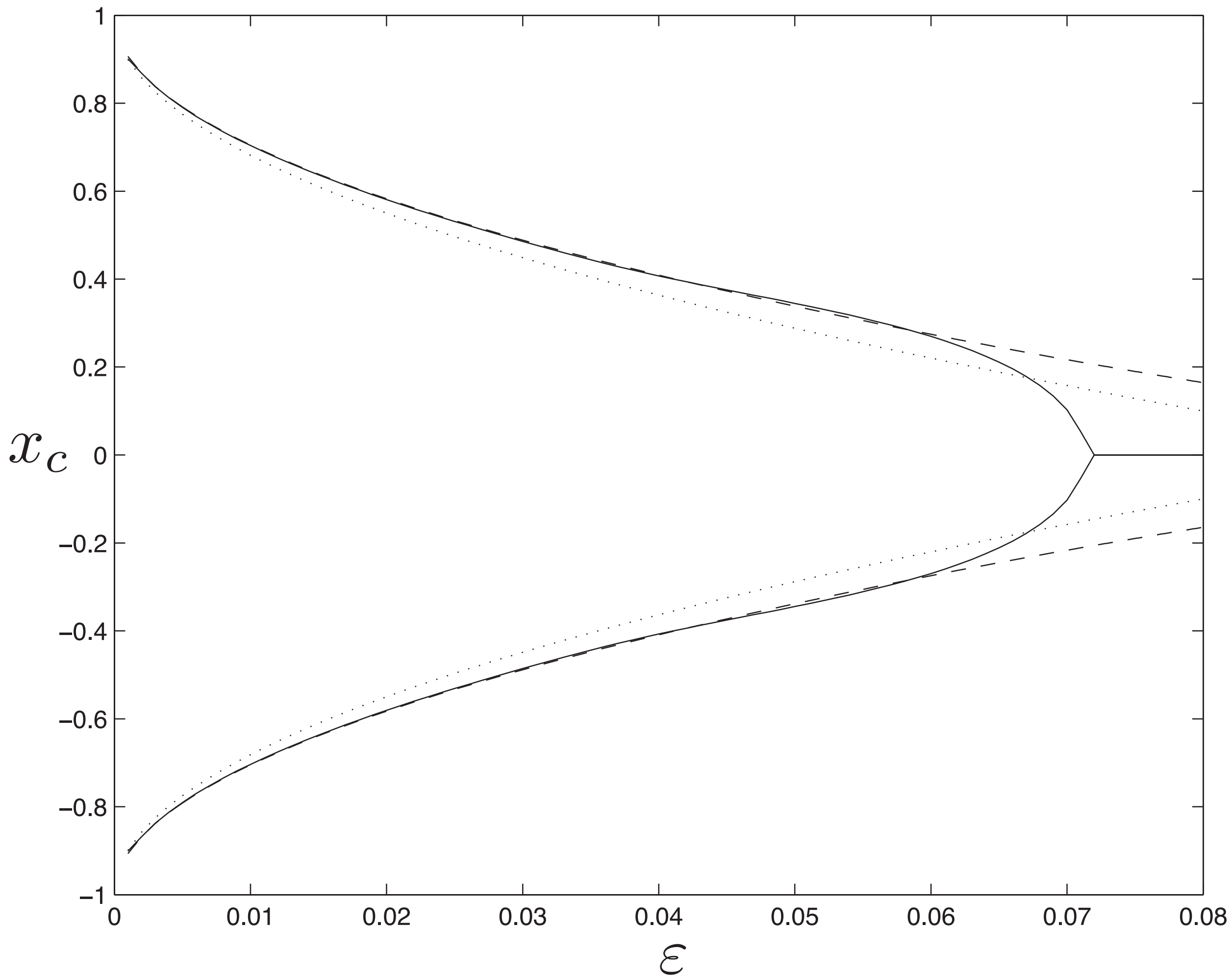
Comparison to full numerics



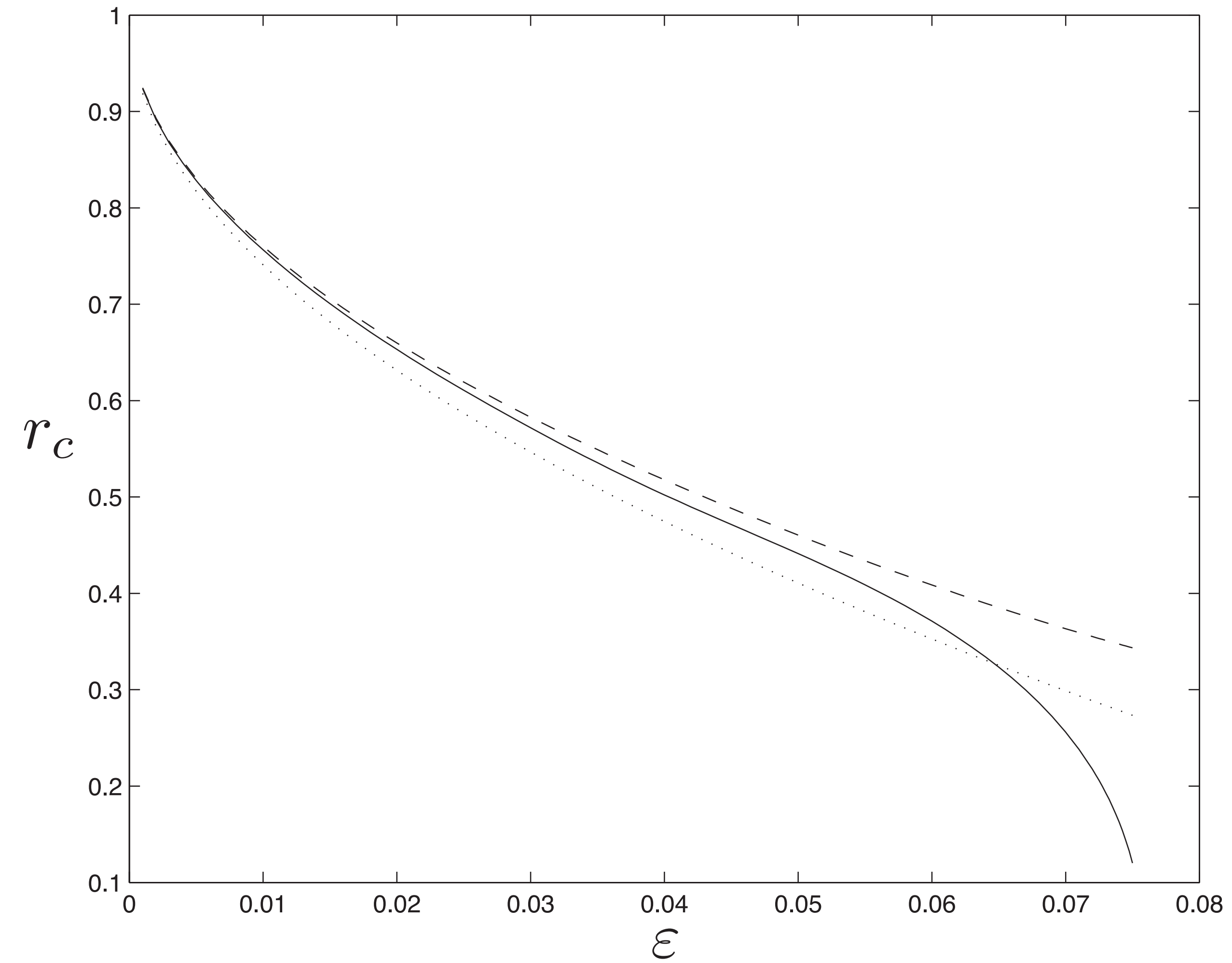
Two time regimes: 1. Short time (Linear), 2. Close to singularity (Nonlinear)

Locating the critical points

- Follow the first critical point of the asymptotic solution.
- Find $v_\eta(\eta_c) = 0, \quad \text{then} \quad x_c = \pm(1 - \varepsilon^{\frac{1}{2}} f(t)^{\frac{1}{4}} \eta_c(t_c))$



1D

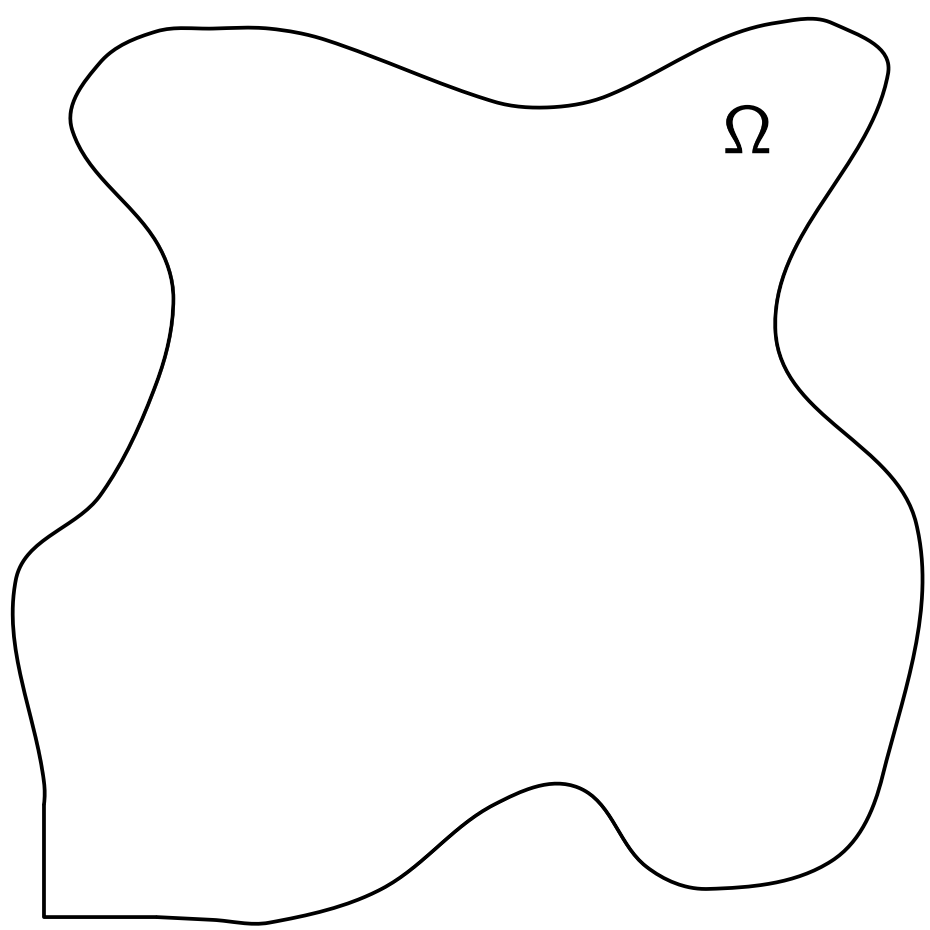


Unit Disk

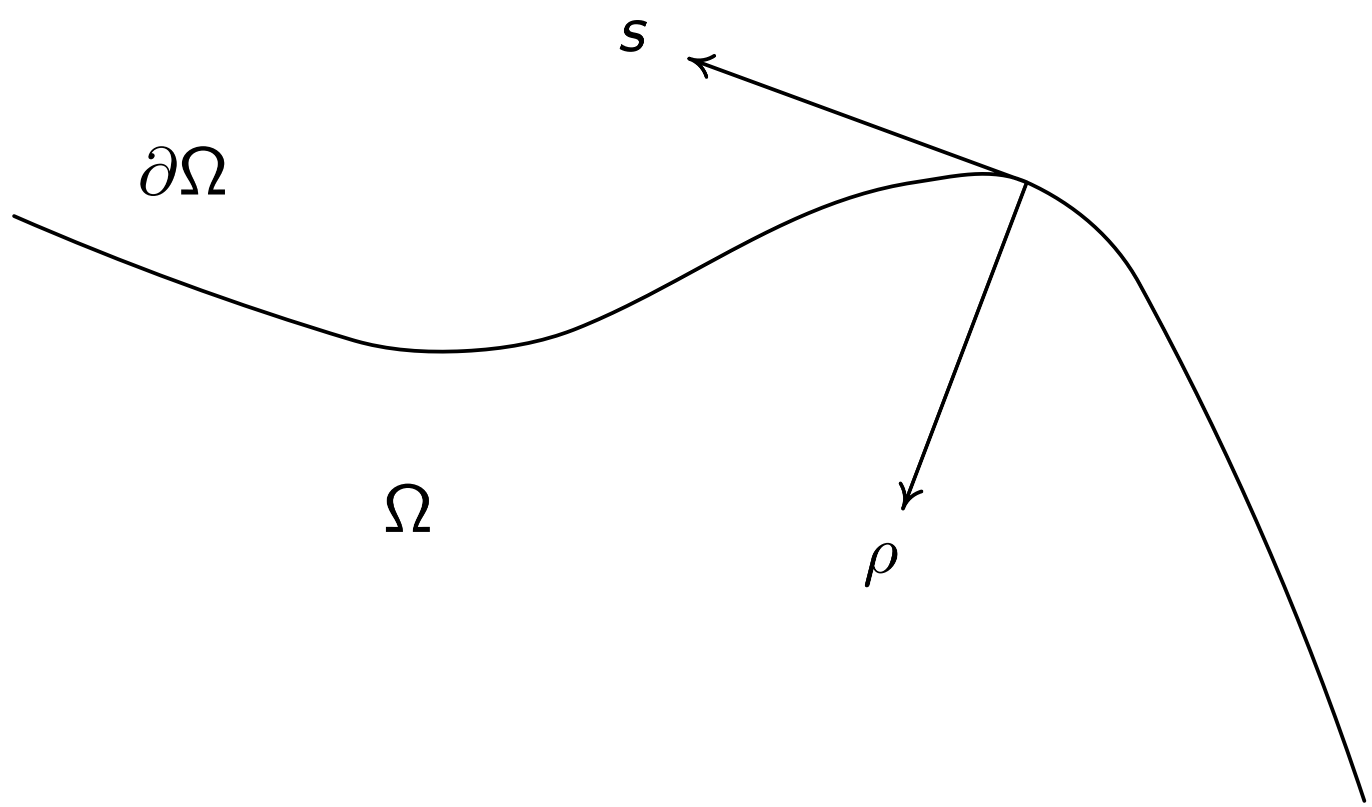
Solid Line (Numerics), Dashed Line (Asymptotics).

Predicting the touchdown set for general geometries in $\mathbb{R}^2$.

- Stability of ring-like touchdown sets.
- What is the touchdown set for more general geometries?
- Is asymmetric touchdown possible?



Asymptotic Breakdown.



$$\Delta \equiv \frac{\partial^2}{\partial \rho^2} - \frac{\kappa(s)}{1-\rho\kappa(s)}\frac{\partial}{\partial \rho} + \frac{\kappa(s)}{1-\rho\kappa(s)}\frac{\partial}{\partial s}\left(\frac{\kappa(s)}{1-\rho\kappa(s)}\frac{\partial}{\partial s}\right)$$

Boundary Layer

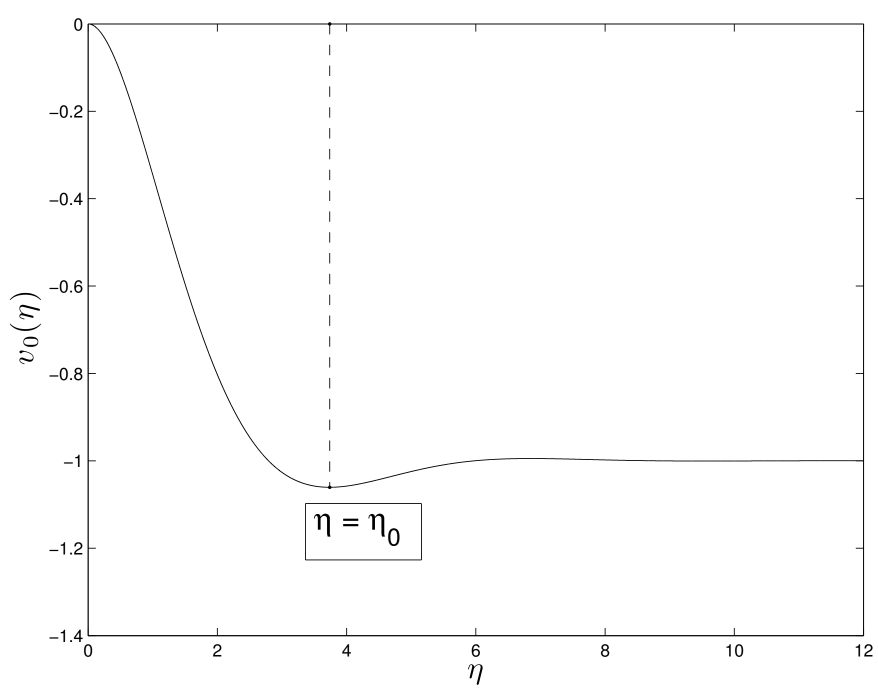
$$\eta = \frac{\rho}{\phi}, \qquad u(x,t) = f(t)v(\eta), \qquad \phi(t) = \varepsilon^{1/2} f(t)^{1/4}$$

Leading order theory:

Same leading order profile:

$$\frac{d^4 v_0}{d\eta^4} - \frac{\eta}{4}\frac{dv_0}{d\eta} + v_0 = -1, \quad \eta > 0; \qquad v_0 \sim -1 \quad \text{as} \quad \eta \to \infty$$

Boundary profile propagates inwards normally to $\partial\Omega$:



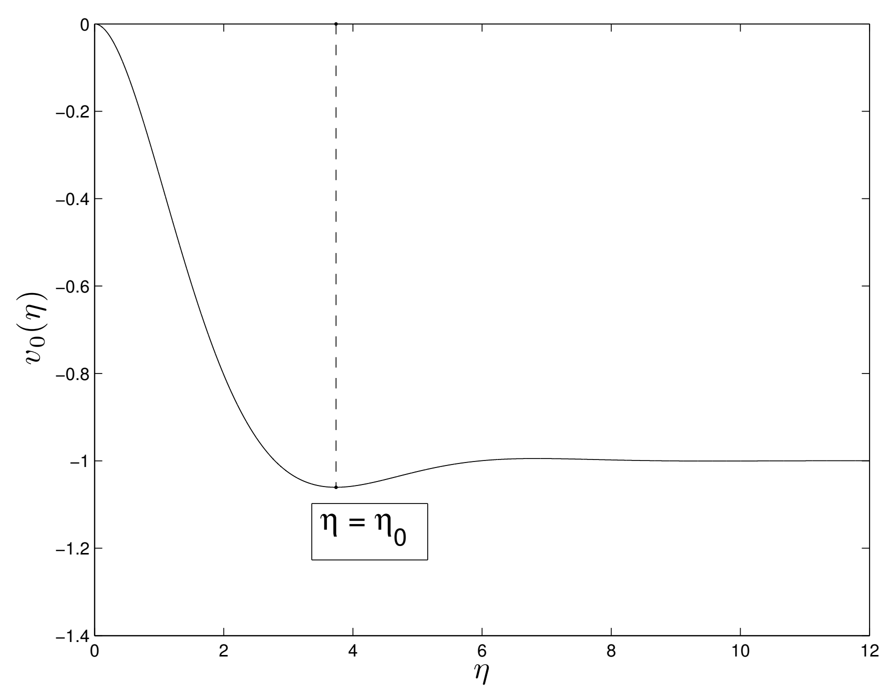
$u(x,t) \sim ?,$ How do we construct a uniform asymptotic solution?

Leading order theory:

Same leading order profile:

$$\frac{d^4 v_0}{d\eta^4} - \frac{\eta}{4}\frac{dv_0}{d\eta} + v_0 = -1, \quad \eta > 0; \qquad v_0 \sim -1 \quad \text{as} \quad \eta \to \infty$$

Boundary profile propagates inwards normally to $\partial\Omega$:

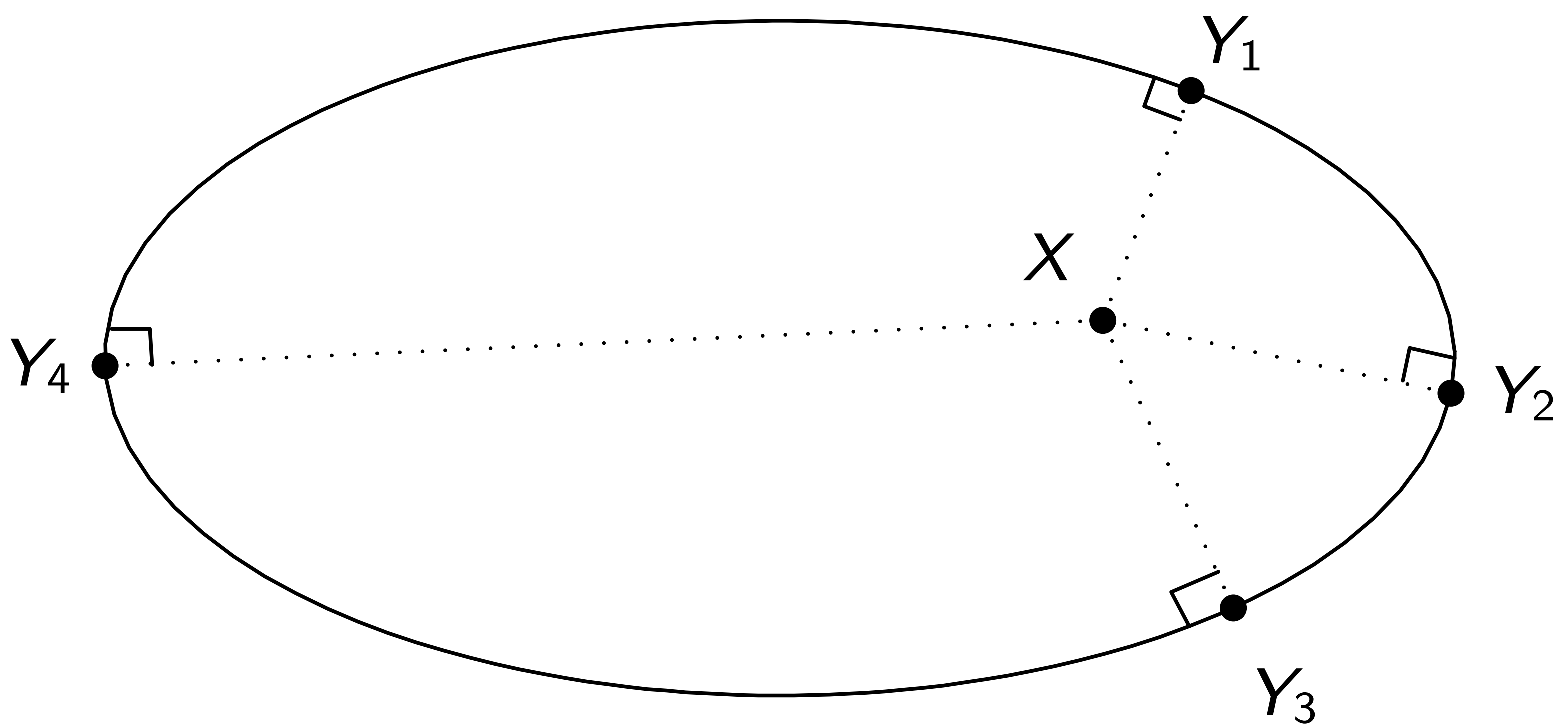


$u(x,t) \sim ?,$ How do we construct a uniform asymptotic solution?

Leading Order Uniform Expansion.

To construct solution at $X \in \Omega$, find all $Y \in \partial\Omega$ satisfying

$$XY \perp \partial_\tau(Y)$$



$$u(X,t) = \underbrace{-f(t)}_{\text{Outer}} + f(t)\left[\sum_{j=1}^{4} \underbrace{v_0\left(\frac{|X-Y_j|}{\phi}\right)}_{\text{Boundary Contributions}} + \underbrace{1}_{\text{Overlap}}\right]$$

Asymptotic reconstruction of profile just before touchdown.

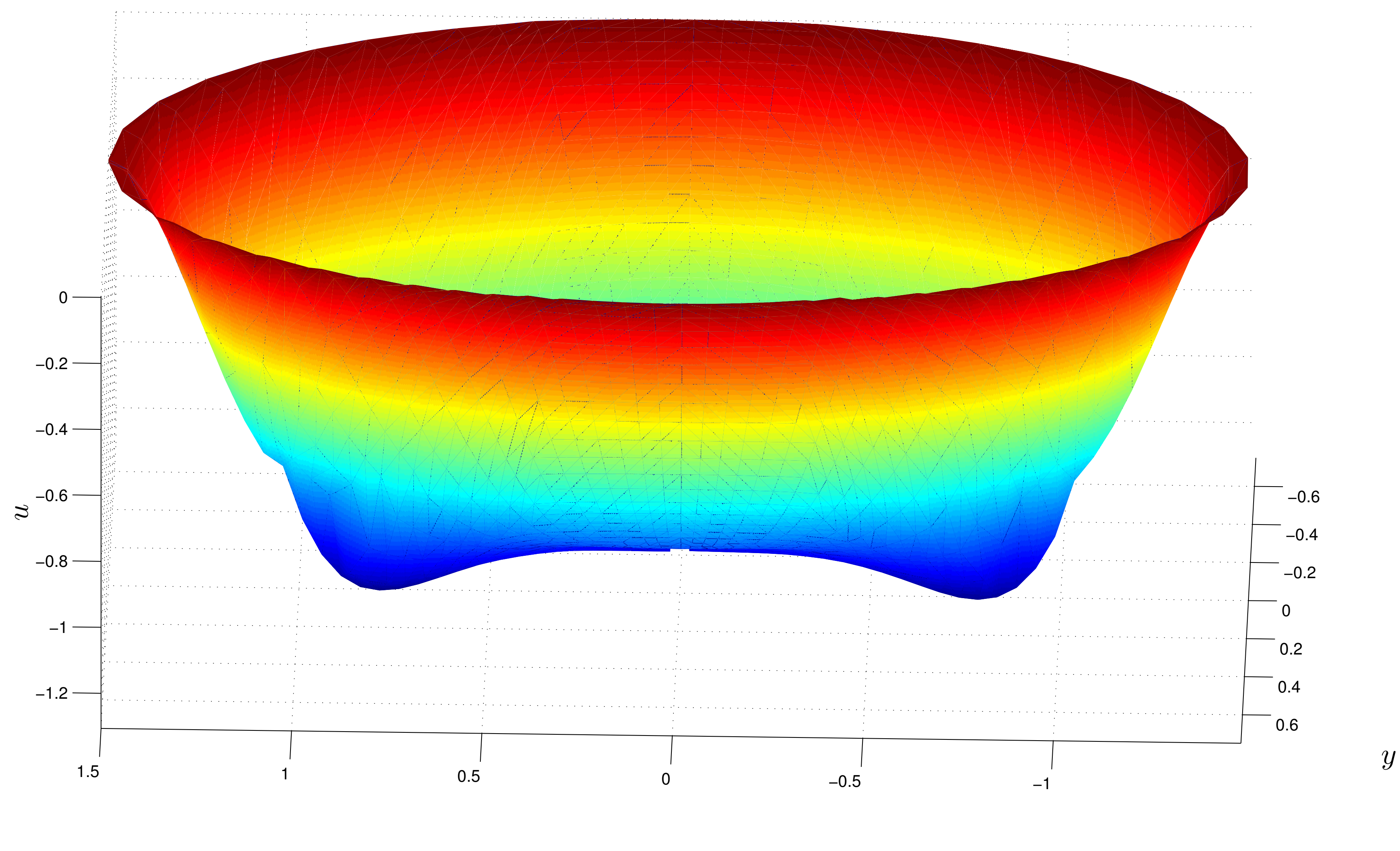
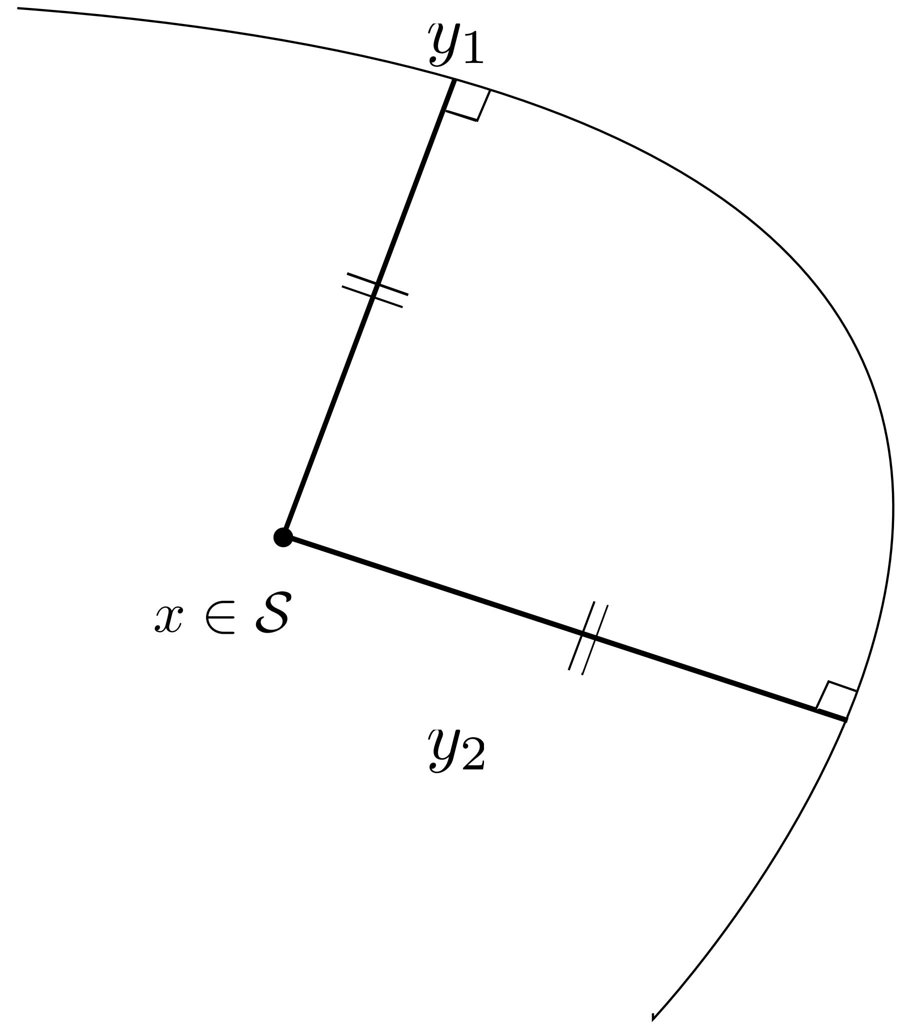


Figure: Ellipse and $\varepsilon = 0.03$.

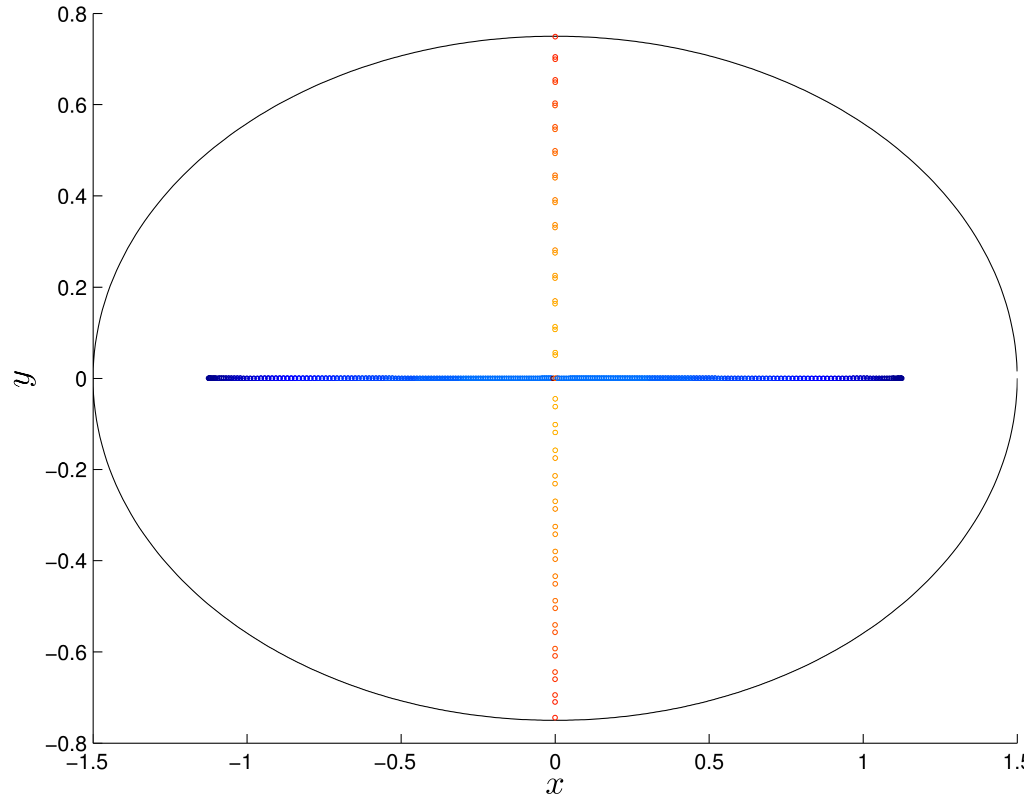
A geometric 'Skeleton Theory' for contact set prediction



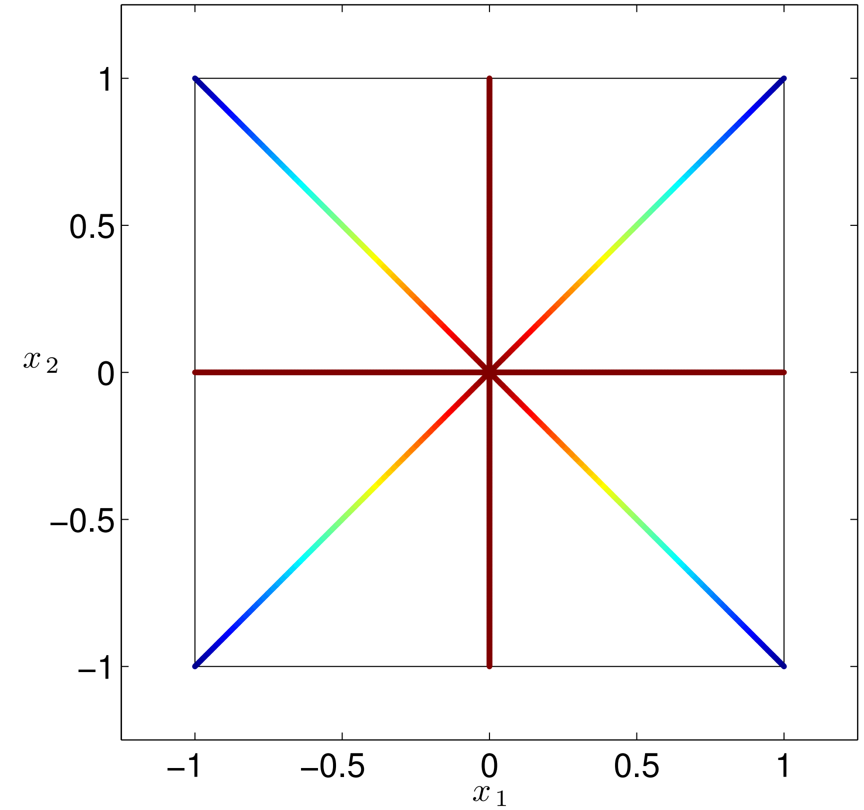
- $x \in \mathcal{S}_\Omega$ if $x \in \Omega$ receives perpendicular contributions from any $y_1, y_2 \in \partial\Omega$ and $|x - y_1| = |x - y_2|$.
- On $\mathcal{S}_\Omega$, troughs superimpose to lower the value of $u(x,t)$.

$$\boxed{\text{Main Idea: At the contact time, minima of asymptotic solution predicts the contact set.}}$$

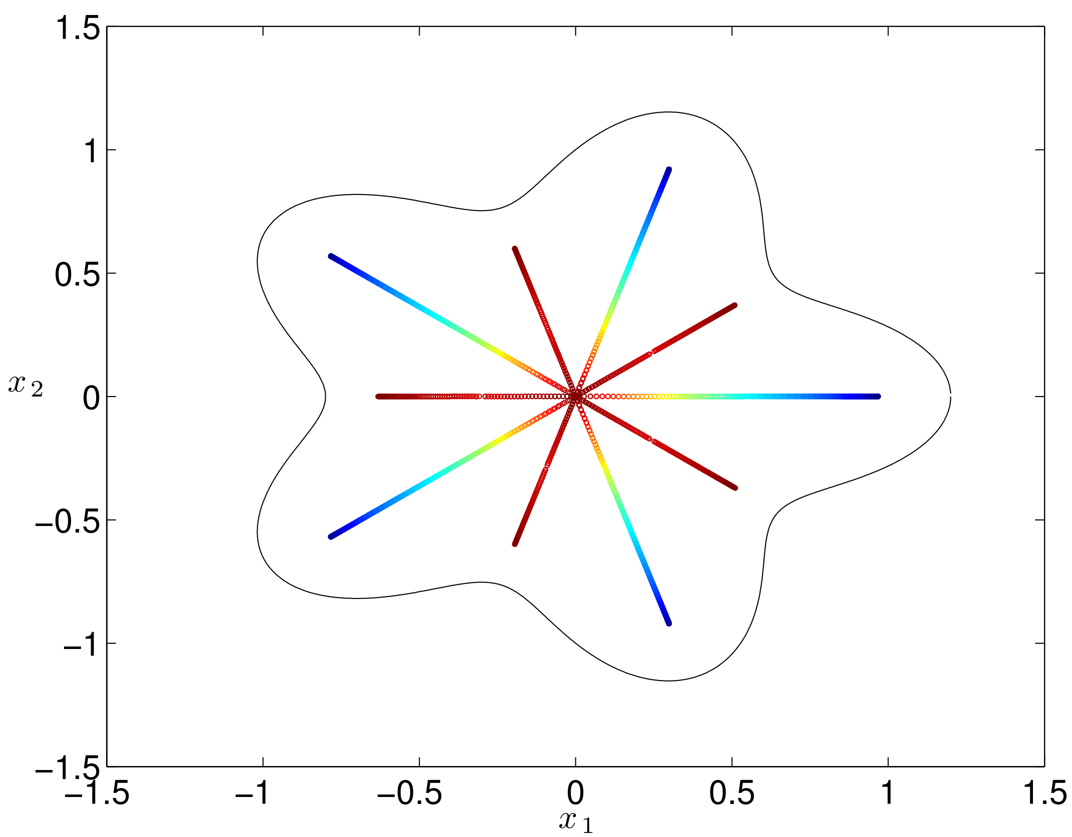
Simple Skeleton Examples

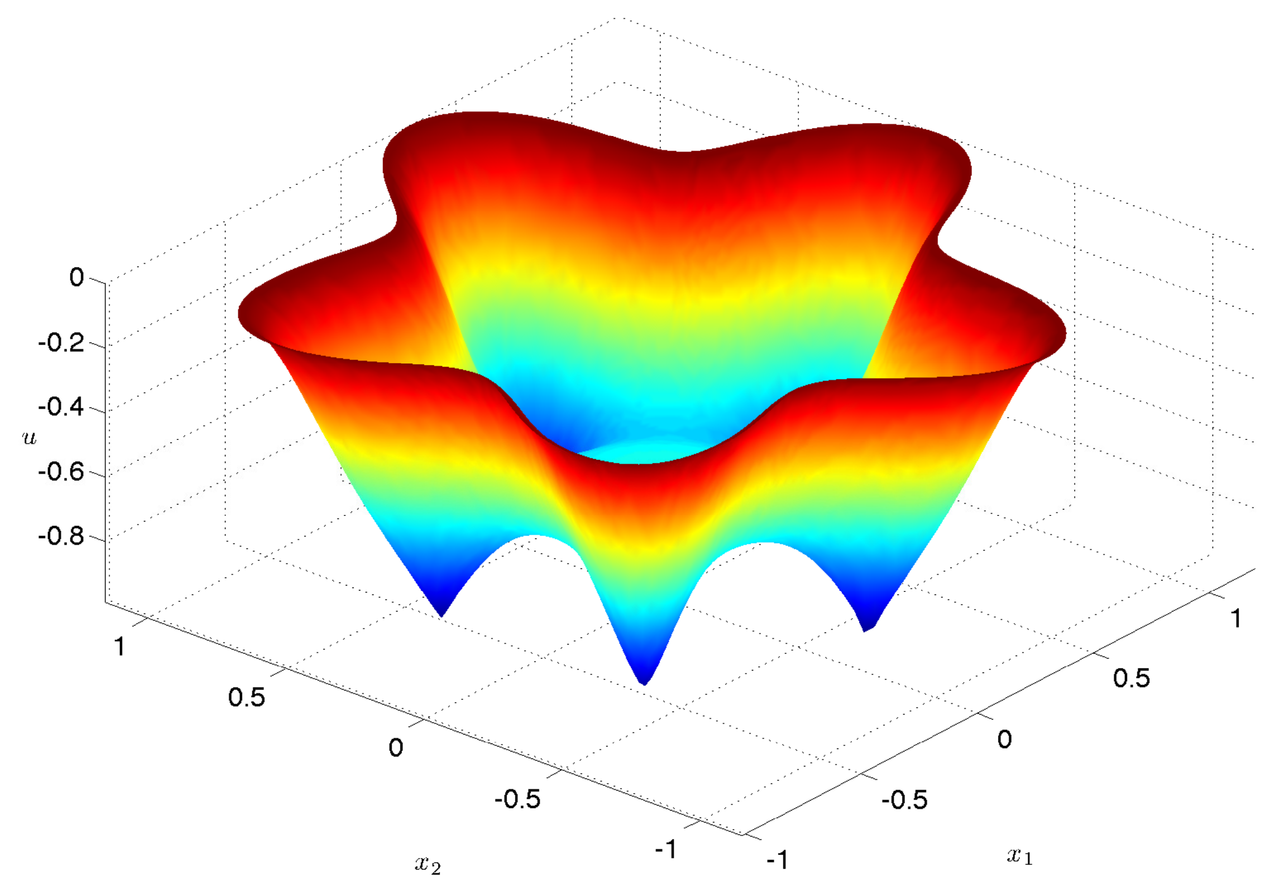


Ellipse

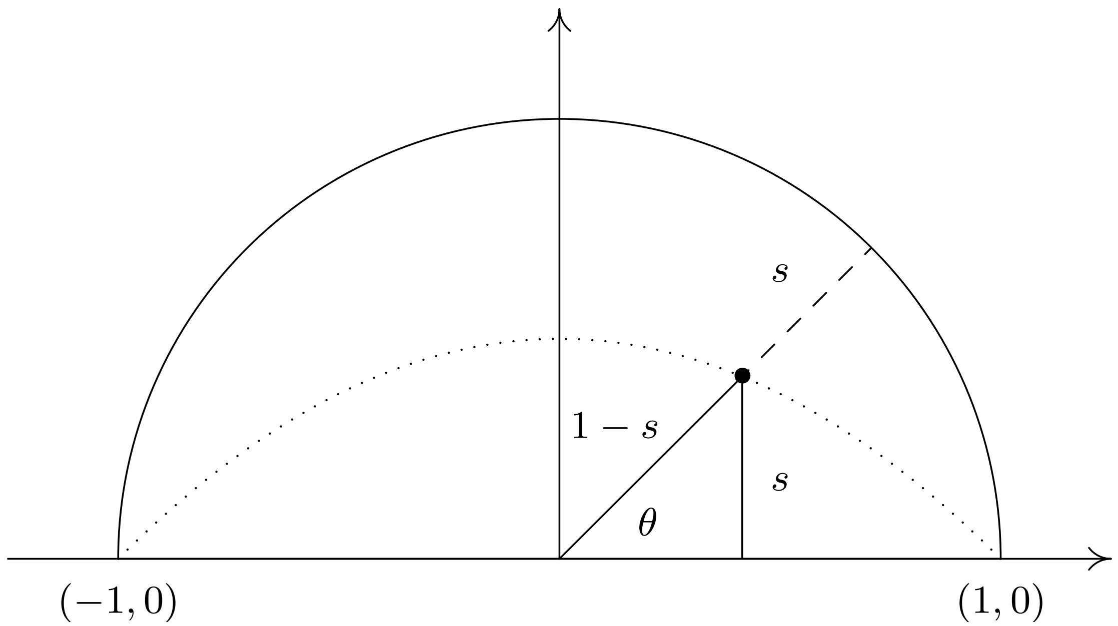


Square





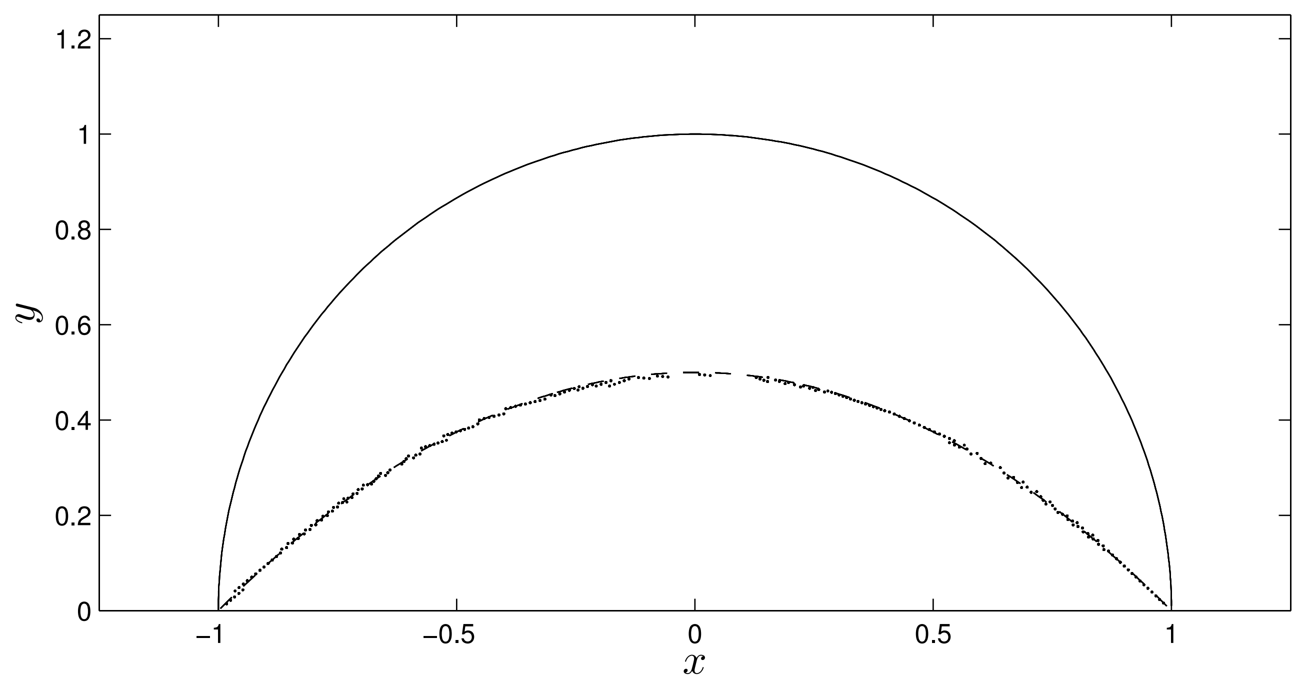
Semicircular Skeleton Examples



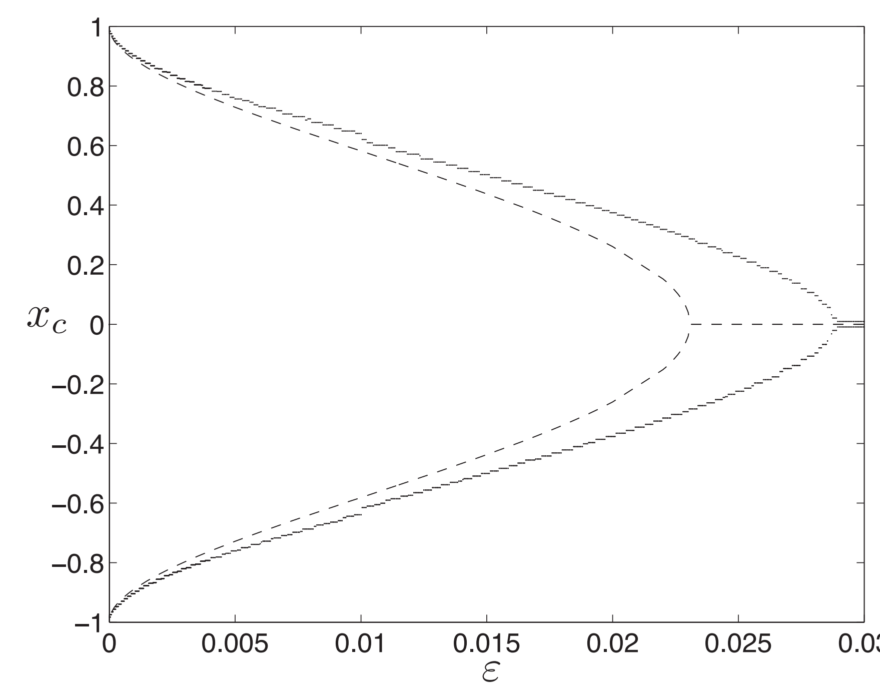
$$\sin\theta = \frac{s}{1-s}$$

$$\mathcal{S}_\Omega = \left\{ \left(\frac{\cos\theta}{1+\sin\theta}, \frac{\sin\theta}{1+\sin\theta} \right),\ \theta \in (0,\pi) \right\}$$

Numerical Comparison



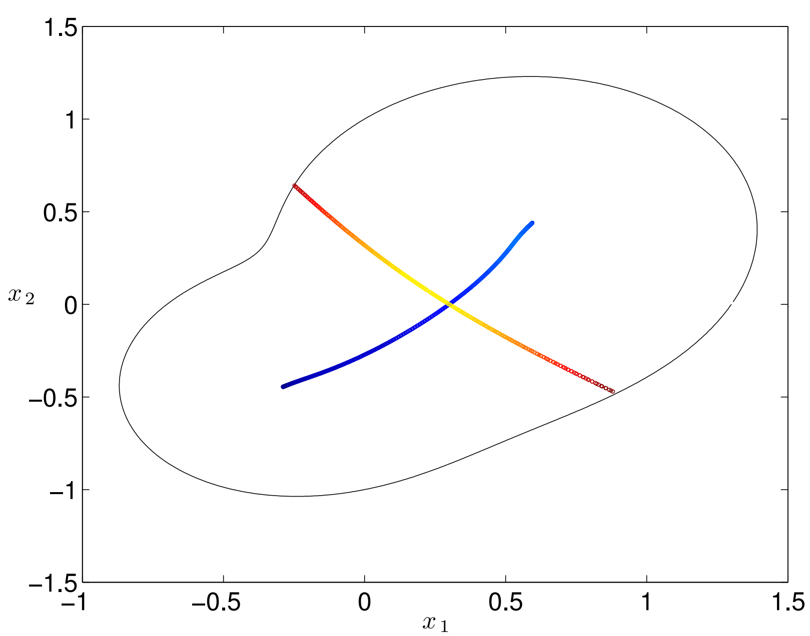
Skeleton captures contact set.



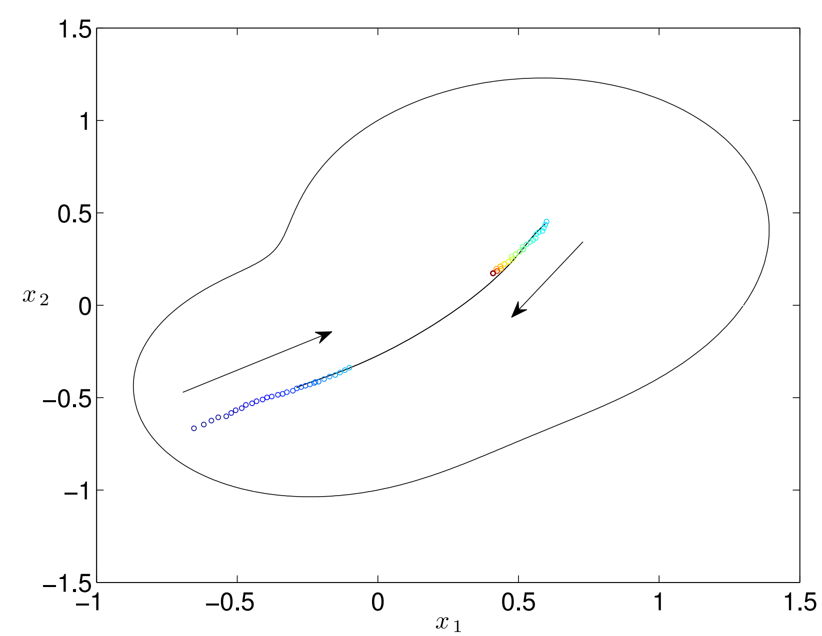
But not for particular parameters

Potato: Example with no symmetries I

$$\partial\Omega = \{(x_1, y_1) = (r(\theta)\cos\theta, r(\theta)\sin\theta) \mid 0 < \theta \leq 2\pi\},$$
$$r(\theta) = 1 + 0.3\,(\cos\theta + \sin 2\theta)$$



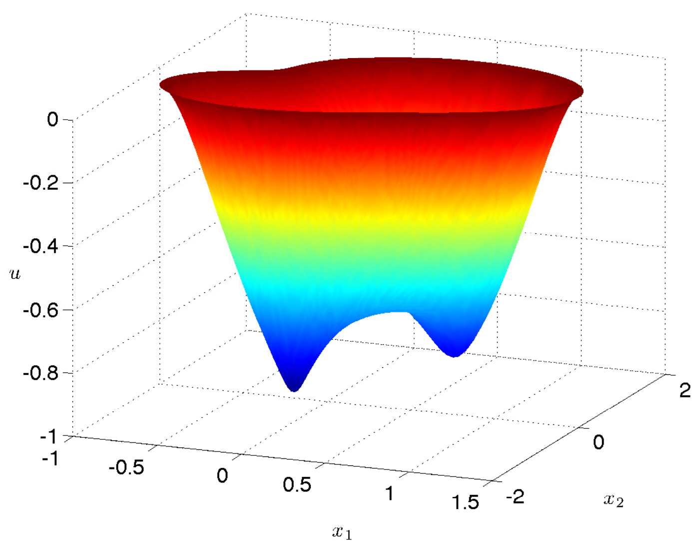
(h) Domain with Full Skeleton



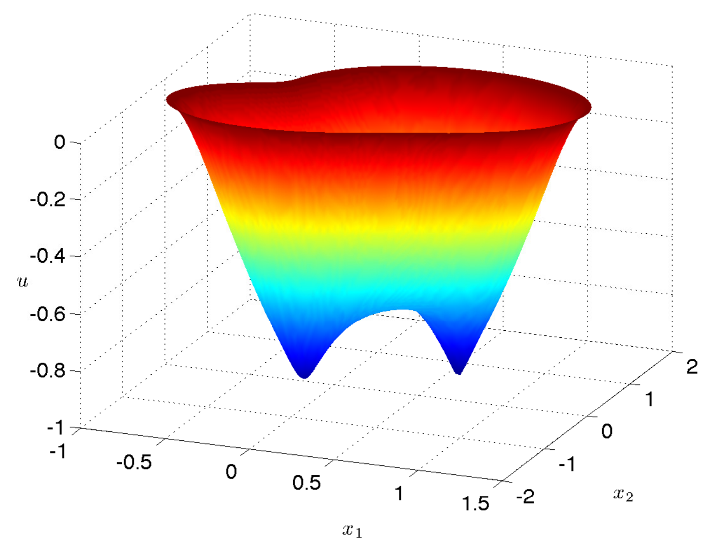
(i) Touchdown points with partial skeleton.

The arrows point in the direction of increasing ε values.

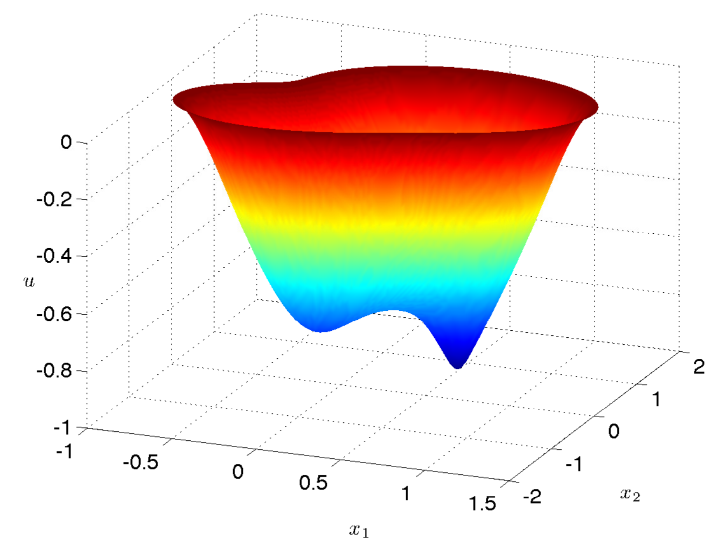
Potato: Example with no symmetries II



(j) $\varepsilon = 0.047$



(k) $\varepsilon = \varepsilon_p \approx 0.04855$



(l) $\varepsilon = 0.055$

- Single point touchdown in left side ($\varepsilon < \varepsilon_p$) and right side ($\varepsilon > \varepsilon_p$).
- For $\varepsilon = \varepsilon_p \approx 0.04855$, touchdown at two points simultaneously.

Multiple singularities generic in blow-up of high order PDEs

$$u_t = -\varepsilon^2 \Delta^2 u + f(u), \quad x \in \Omega; \qquad u = \partial_n u = 0, \quad x \in \partial\Omega.$$

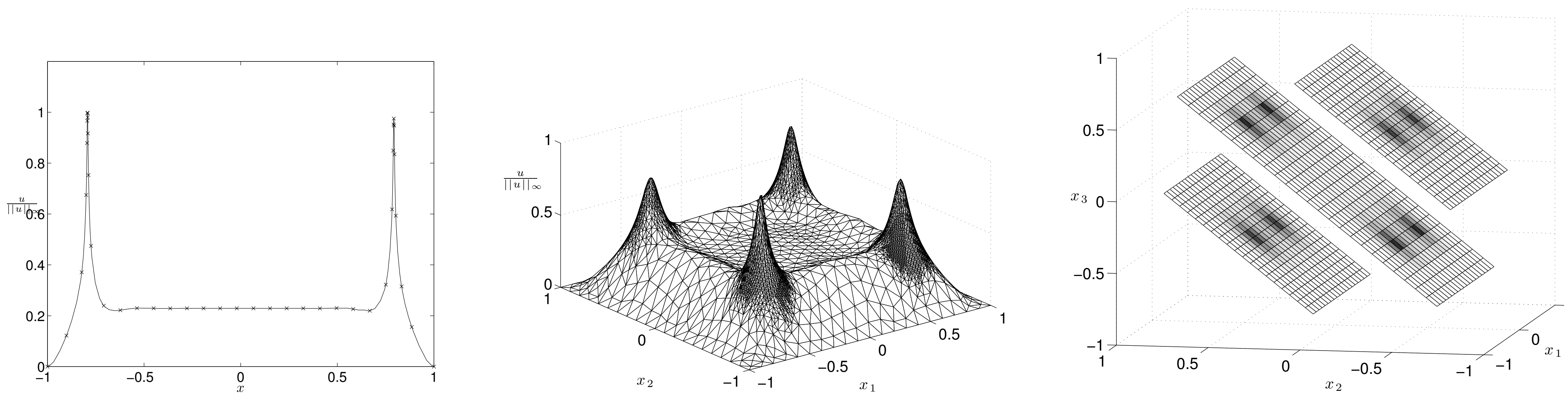


Figure: $f(u) = e^u$, $\Omega = [-1, 1]^n$ for $n = 1, 2, 3$.

Second Order Problem

$$u_t = \varepsilon^2 \Delta u + f(u), \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega;$$

$$x_c \sim \max_{x \in \Omega} d(x, \partial\Omega).$$

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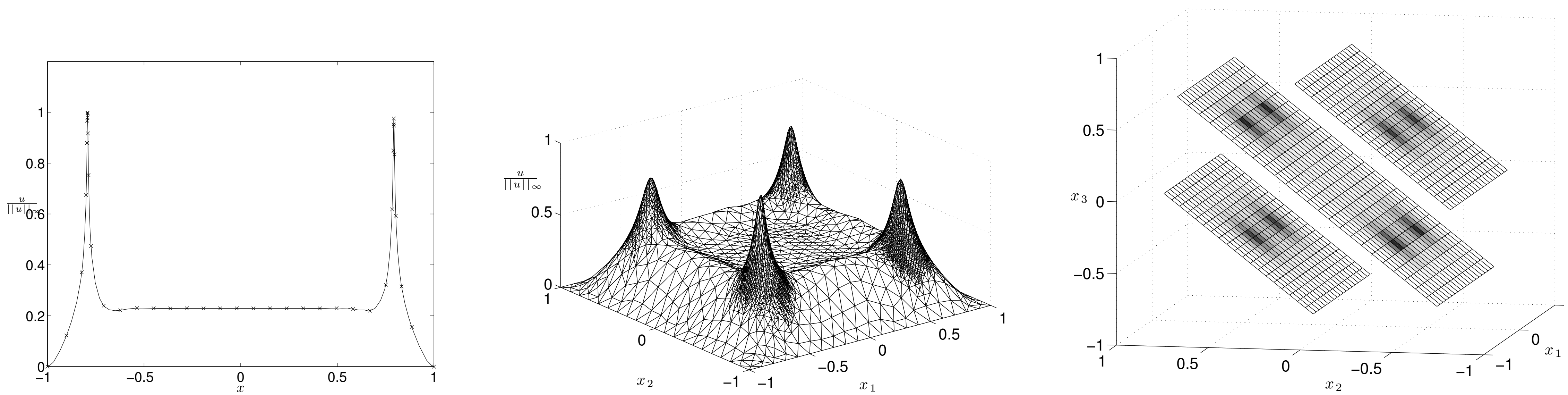

Figure: $f(u) = e^u$, $\Omega = [-1,1]^n$ for $n = 1, 2, 3$.

Second Order Problem

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Outline of Talk

1. Adaptive numerical methods.

   - r-adaptive meshes for generating meshes.
   - Meshes inherit the symmetries/scaling properties of the PDE.

2. Predicting the set of contacts.

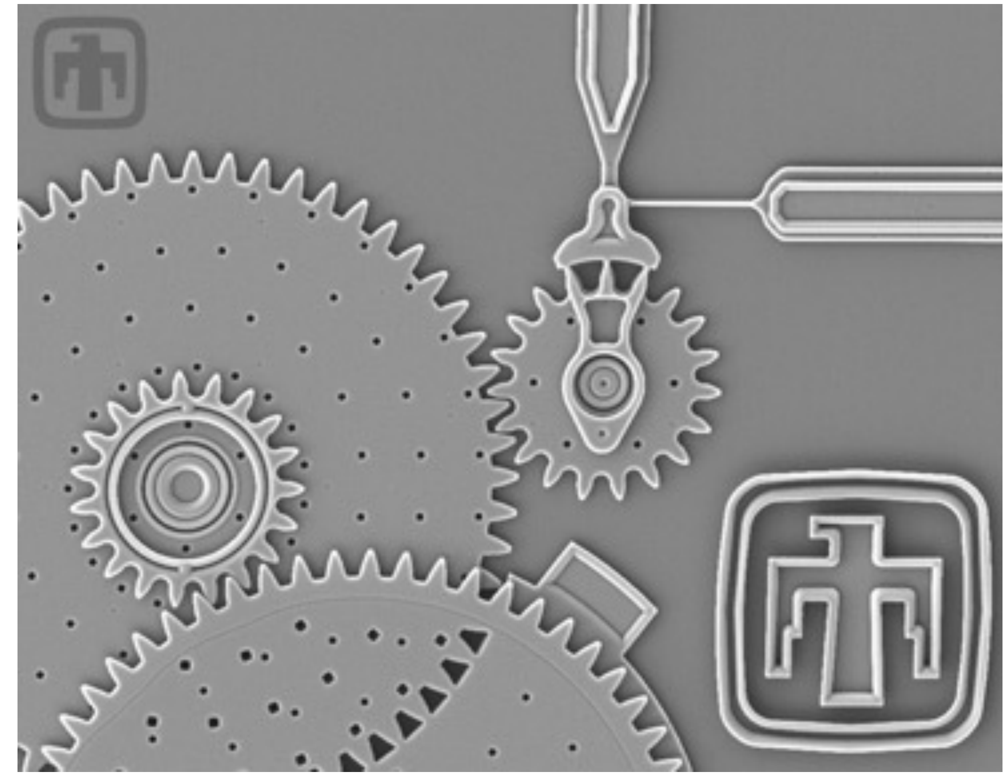
   - Concept set complexity described by a boundary layer analysis.
   - Prediction of contact sets in 1D and general 2D regions.

3. Regularized problem describing post contact dynamics.

   - How do we make sense of solutions beyond initial singularities?
   - Layer dynamics and numerical simulations of sharp interfaces.

Stiction and Adhesion in MEMS

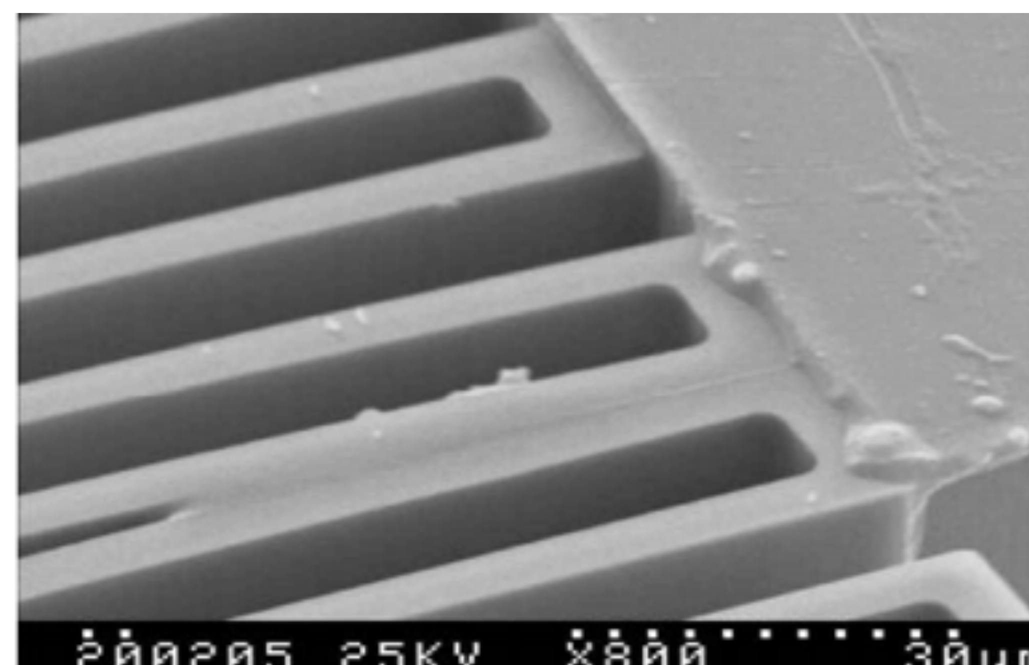
Contact between surfaces in MEMS allows for extended operating regimes.



Ref: sandia.gov

Additional physics once surfaces have come into physical contact

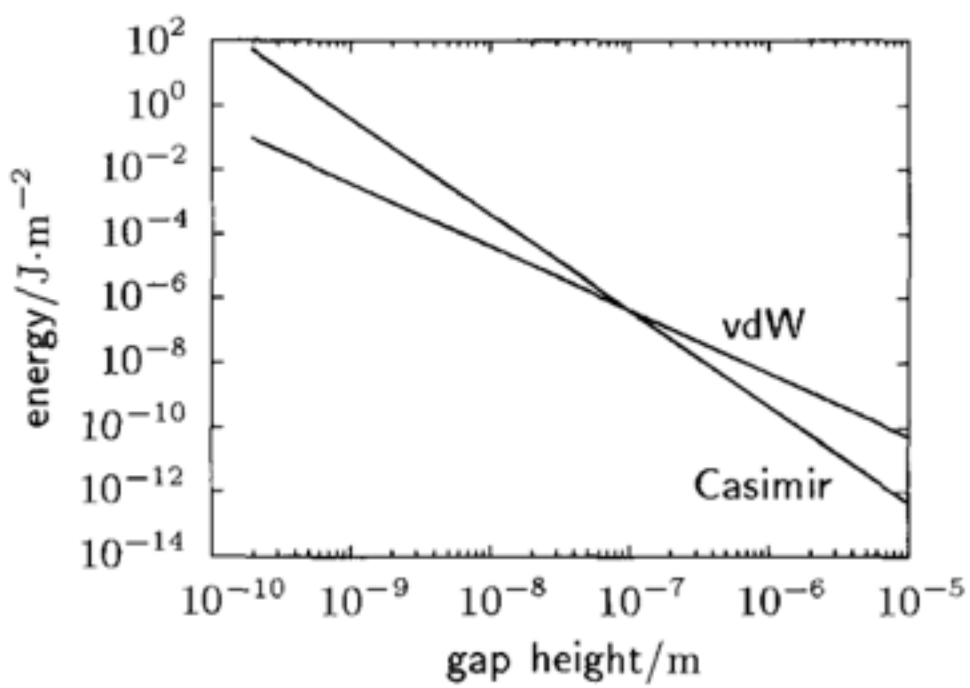
- Frictional forces
- Van der Waal forces
- Casimir effect
- Plasticity of elastic components



J. Adhesion Sci. Tech 17(4) pp 519-546.

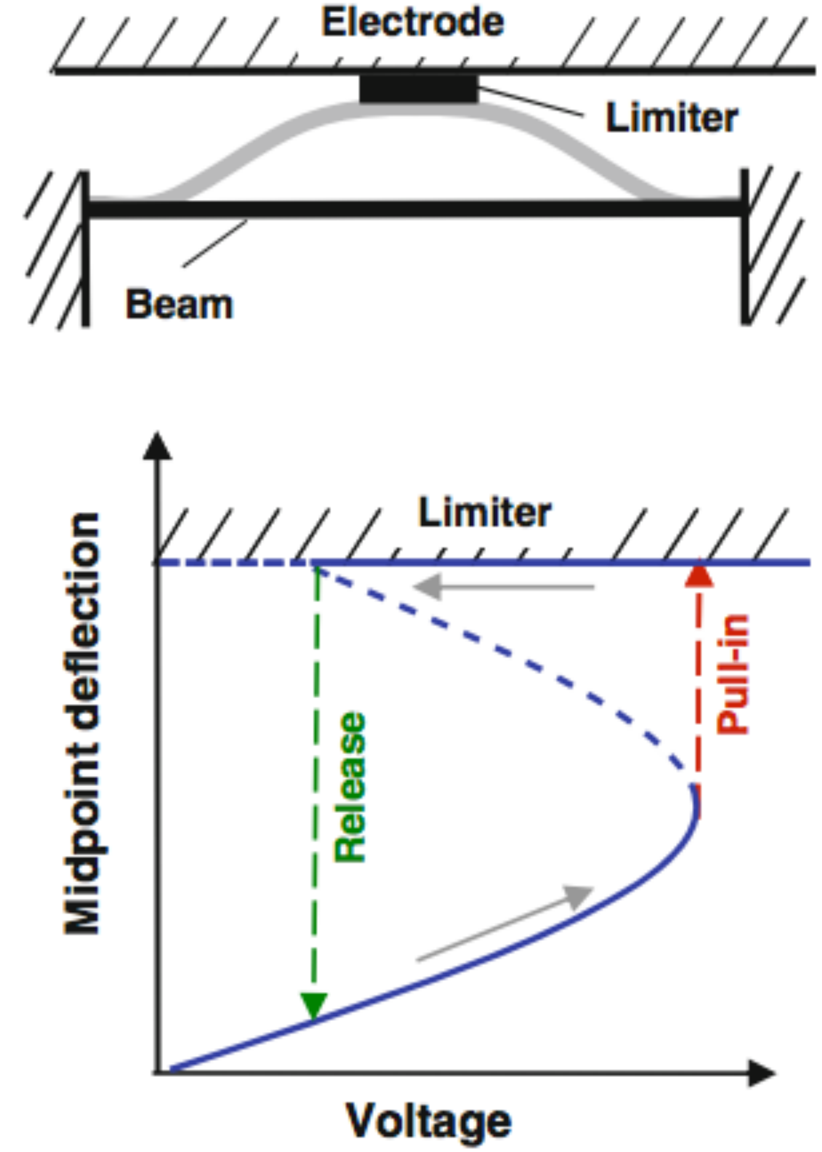
Physical effects at very small gap spacing

$$
\begin{aligned}
F_{\text{casimir}} &= -\frac{\pi^2 \hbar c}{240 d^4} \\
F_{\text{VdW}} &= -0.28 \frac{\omega_p \hbar c}{16\pi\sqrt{2} d^3}
\end{aligned}
$$



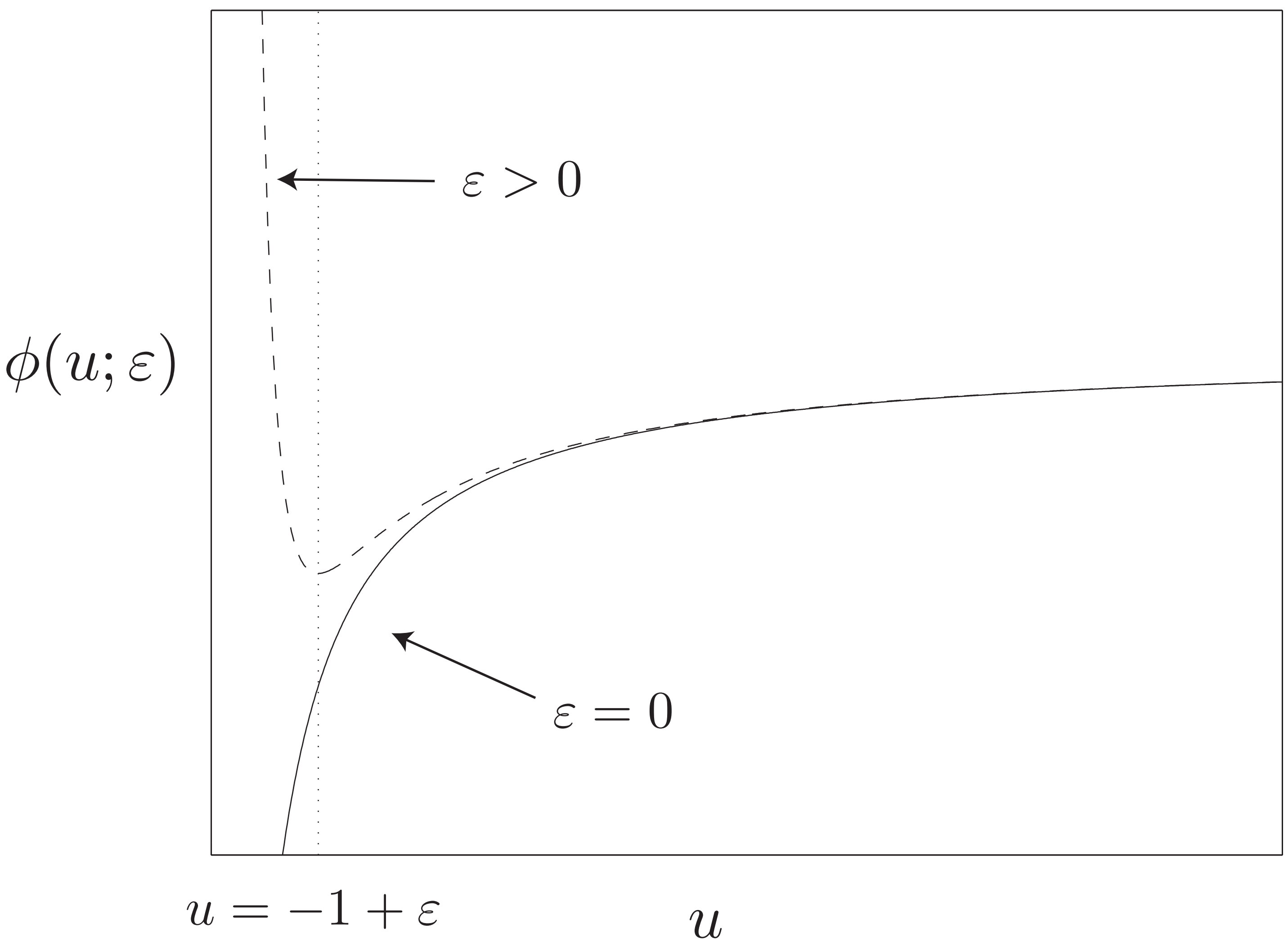
Ref: Acta Mechanica Sinica 19(1).

Physical Barrier



Ref: Krylov, Dick,
Continuum Mech Thermodyn, 22 pp.445-468.

- Ex: $(1+u) \geq \varepsilon\phi(x) > \varepsilon$
- "Obstacle Problem".



Repulsive for $(1+u) < \varepsilon$

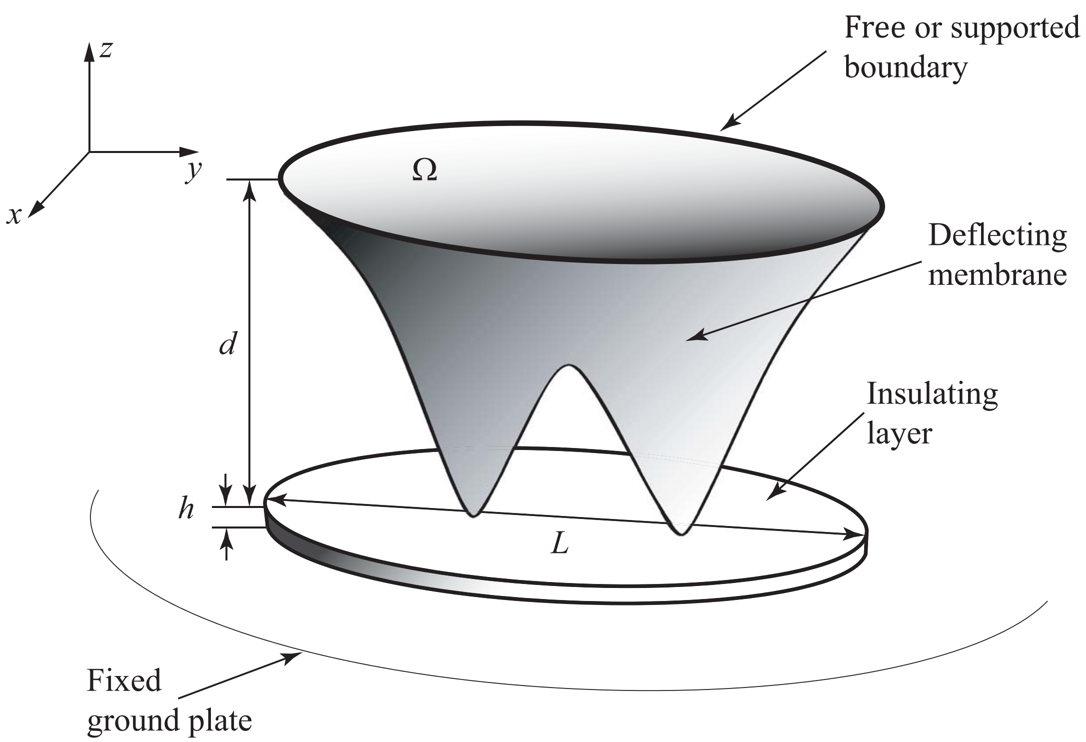
Attracting for $(1+u) > \varepsilon$

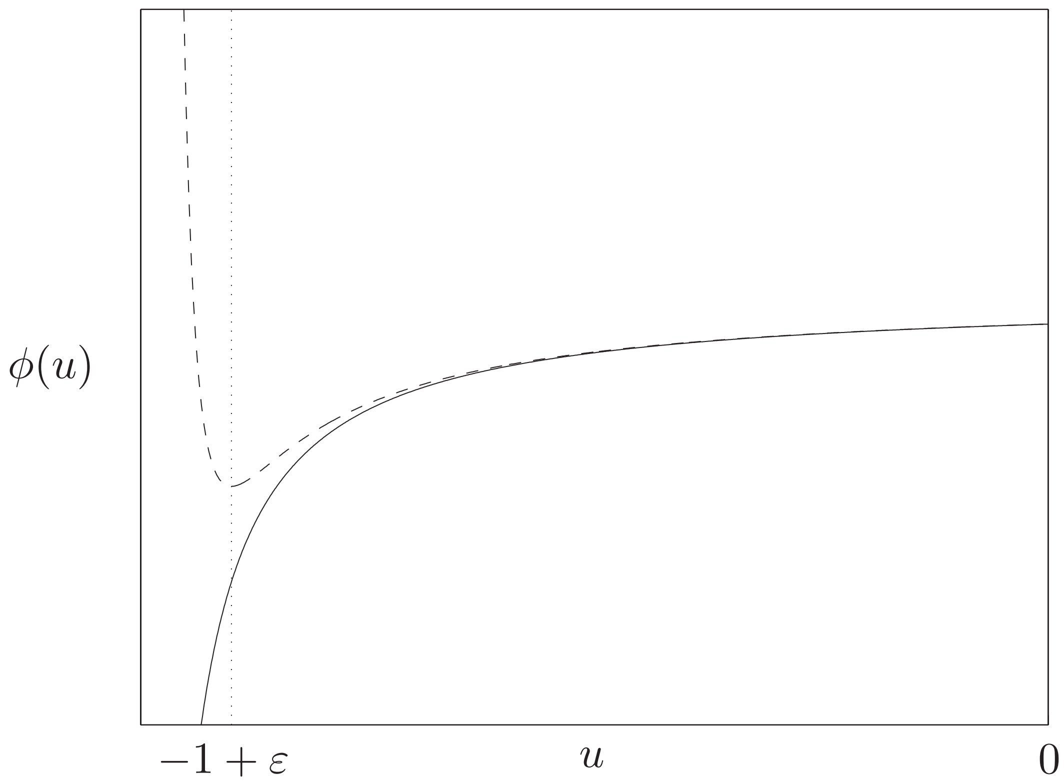
Lennard-Jones type potential

$$\phi(u;\varepsilon) = -\frac{\lambda}{(1+u)} - \frac{\lambda\varepsilon^{m-2}}{(m-1)(1+u)^{m-1}}, \qquad m \geq 3$$

m=3: Van der Waal forces. m=4: Casimir forces.

Regularization of touchdown.





Perturbed parabolic PDEs

$\varepsilon \ll 1$ - Small Regularizing Parameter

Second Order Regularized Model

$$\frac{\partial u}{\partial t} = \Delta u - \frac{\lambda}{(1+u)^2} + \frac{\lambda \varepsilon^{m-2}}{(1+u)^m}, \qquad x \in \Omega; \qquad u = 0 \quad x \in \partial\Omega$$

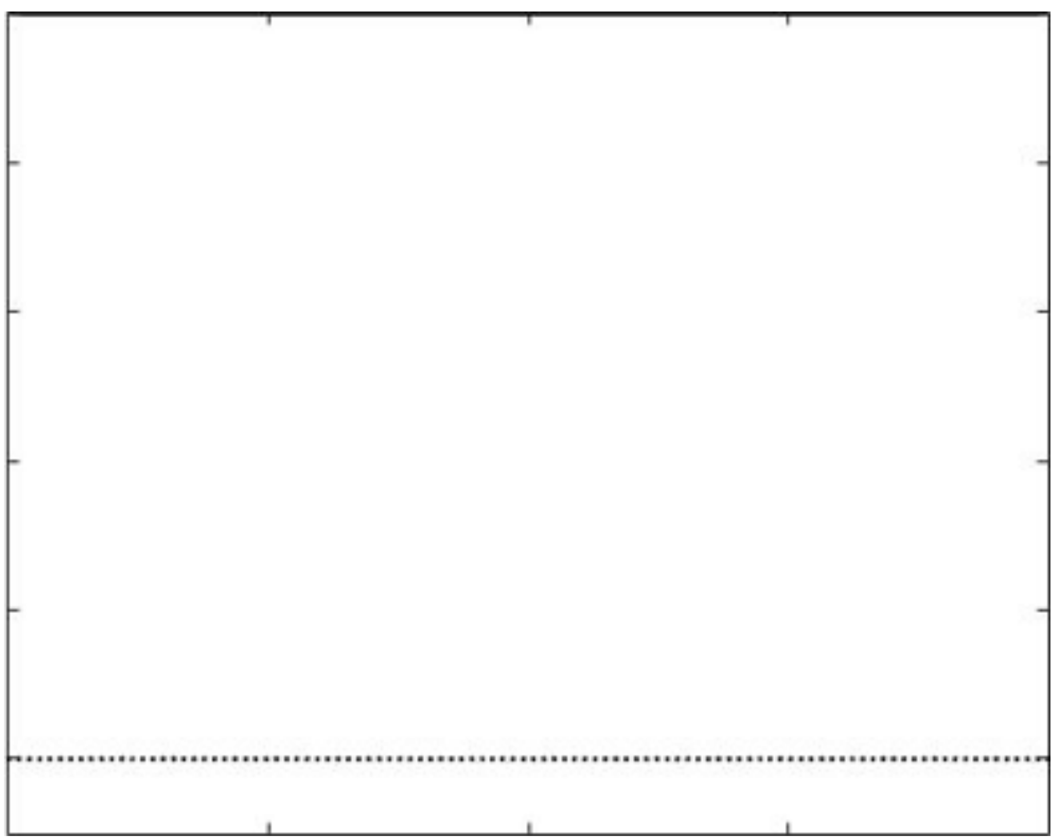
Fourth order Regularized Model

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \frac{\lambda}{(1+u)^2} + \frac{\lambda \varepsilon^{m-2}}{(1+u)^m}, \quad x \in \Omega; \qquad u = \partial_n u = 0, \quad x \in \partial\Omega.$$

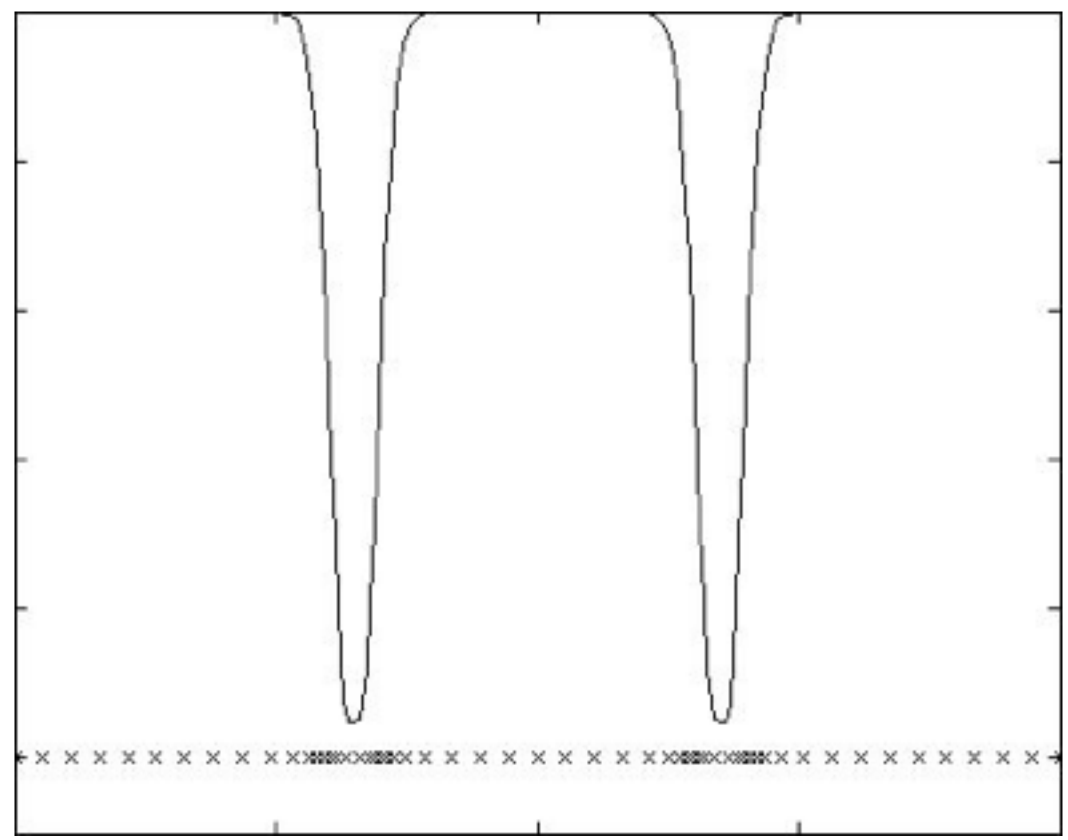
Global Existence

Theorem 1: (Global existence - Laplacian)
Suppose $u_0 \in C^0(\Omega)$ and $u_0 > -1$.
Then $u(x,t) > \min(\inf u_0, -1+\varepsilon)$

Theorem 2: (Global existence - Bi-Laplacian)
Suppose $u_0 \in H^2(\Omega) \cap C^0(\Omega)$ and $u_0 > -1$.
Then $u(x,t)$ exists for all $t > 0$, provided $m \geq 3$ when $\dim(\Omega) = 1$ and $m > 3$ when $\dim(\Omega) = 2$.

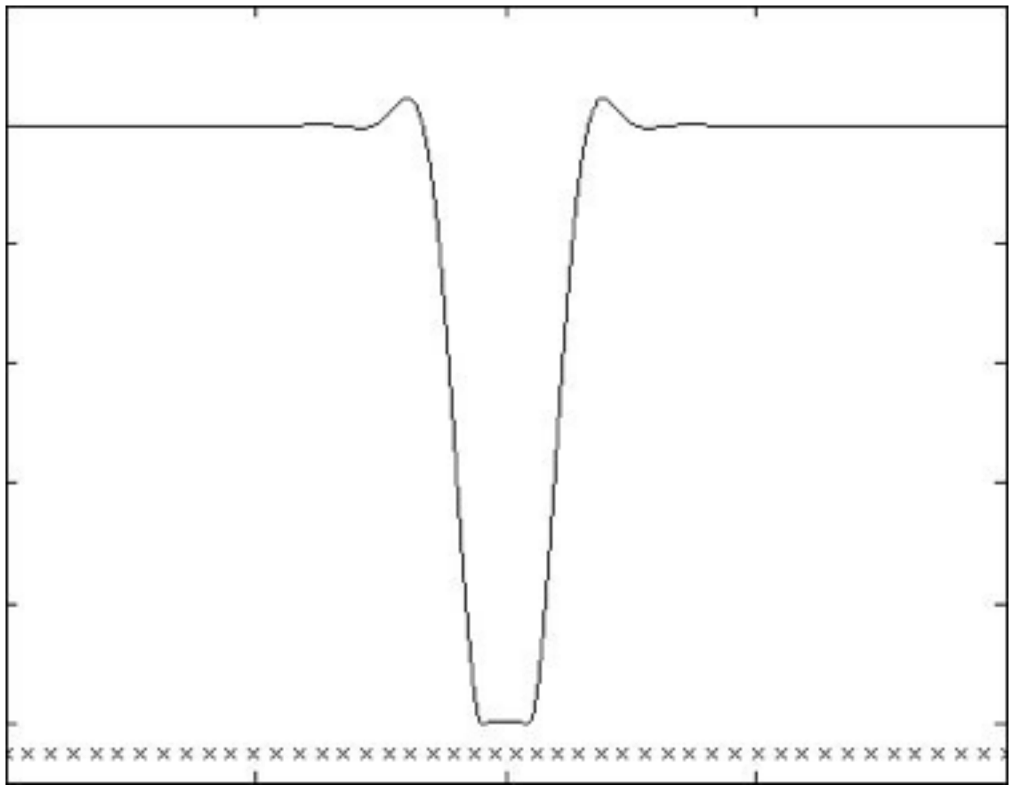


Laplacian: $\lambda = 4, \quad \varepsilon = 0.05$

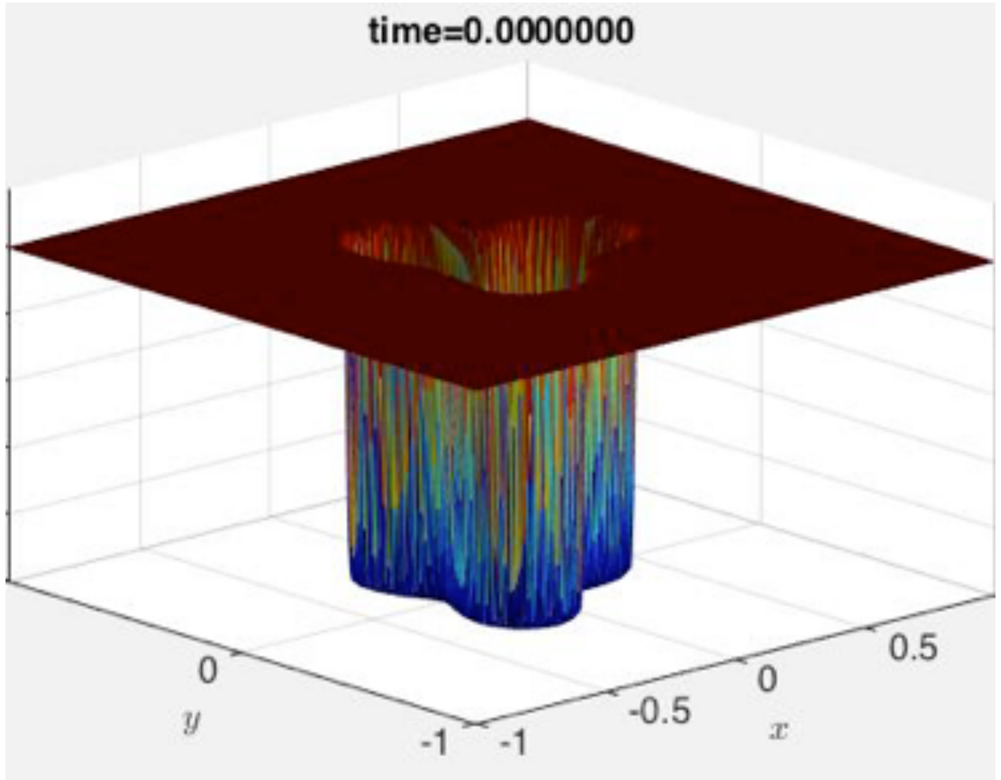


Laplacian: $\lambda = 20, \quad \varepsilon = 0.05$

Fourth order and two dimensional simulations

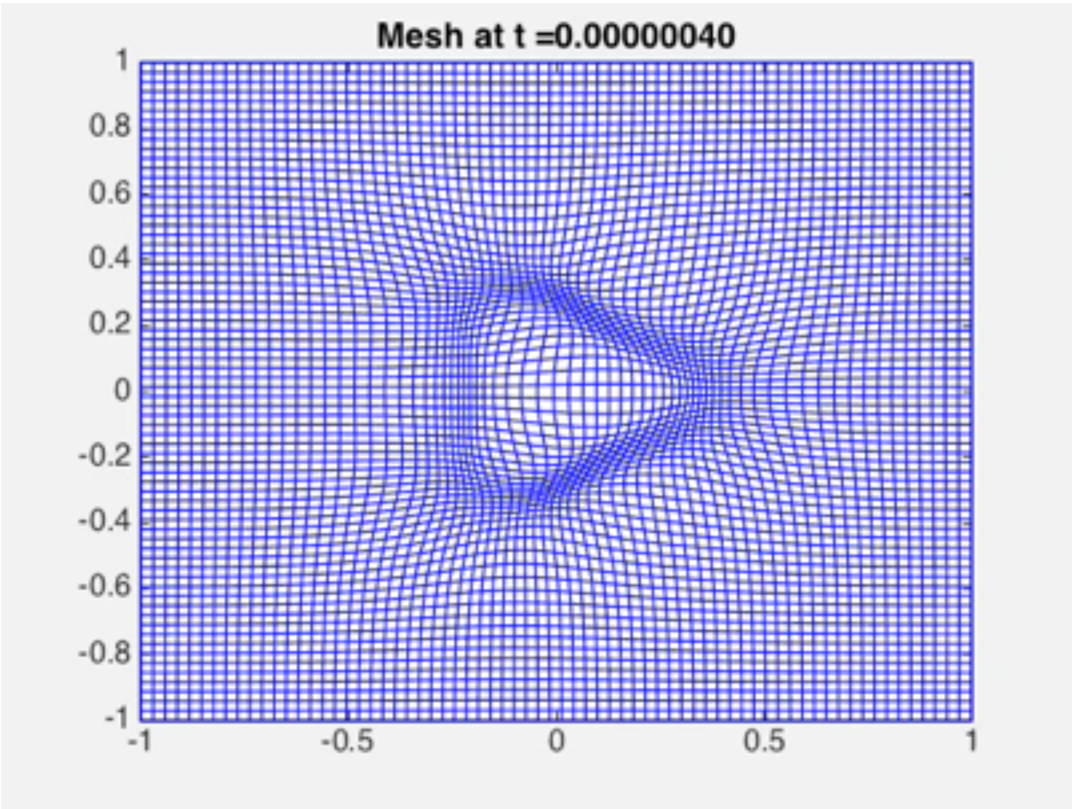


Bi-Laplacian 2D: $\lambda = 200, \quad \varepsilon = 0.005$



Bi-Laplacian 2D: $\lambda = 200, \quad \varepsilon = 0.005$

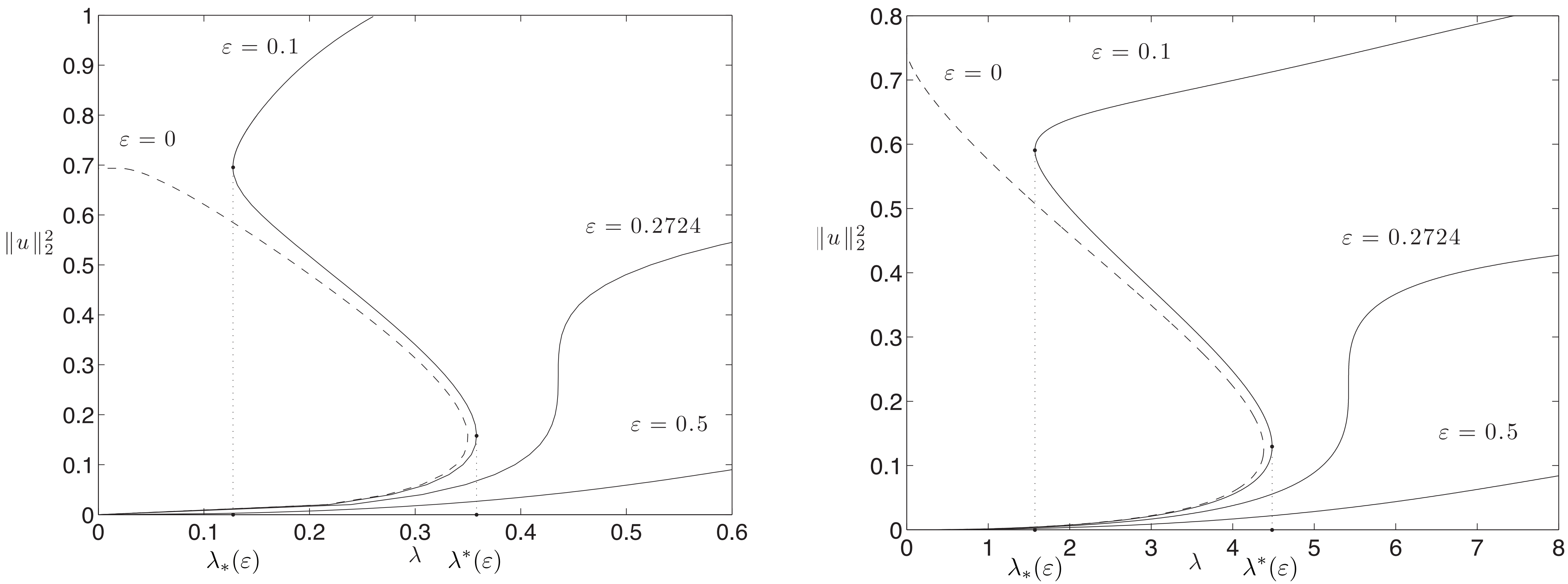
- Sharp interface is oscillatory in both cases.
- In 2D, the amplitude oscillations are modulated by the curvature of the interface.



Bifurcation Diagrams

$$-u_{xxxx} = \frac{\lambda}{(1+u)^2} - \frac{\lambda\varepsilon^{m-2}}{(1+u)^m}, \quad x \in (-1,1); \qquad u = u_x = 0, \quad x = \pm 1$$

$$u_{xx} = \frac{\lambda}{(1+u)^2} - \frac{\lambda\varepsilon^{m-2}}{(1+u)^m}, \quad x \in (-1,1); \qquad u = 0, \quad x = \pm 1$$



- New branch of stable large norm equilibria emerges after second fold point.
- <u>Bistability</u> possible from switching between large and small norm solutions.

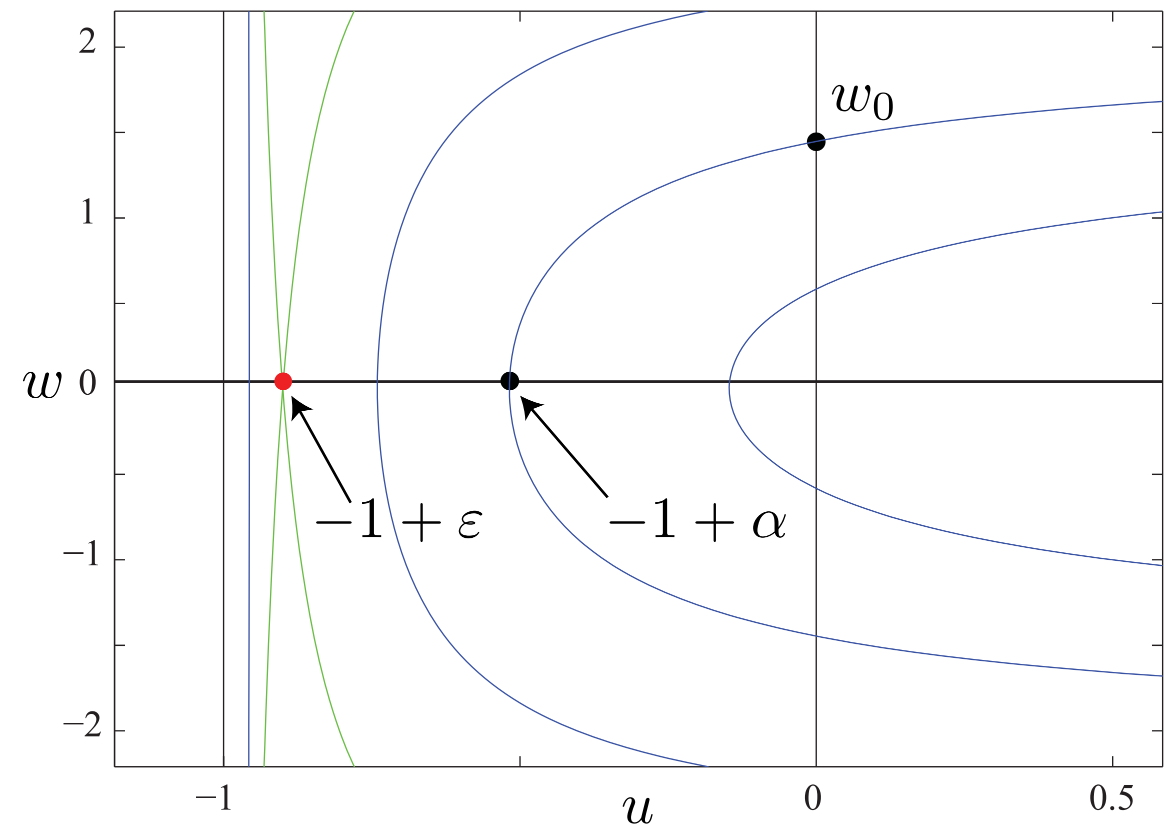
Equilibrium Analysis in Laplacian Case.

$$y = \sqrt{\lambda}x \quad \Longrightarrow \quad u_{yy} = \frac{1}{(1+u)^2} - \frac{\varepsilon^{m-2}}{(1+u)^m}, \quad y \in [-\sqrt{\lambda}, \sqrt{\lambda}], \quad u(\pm\sqrt{\lambda}) = 0$$

Autonomous System

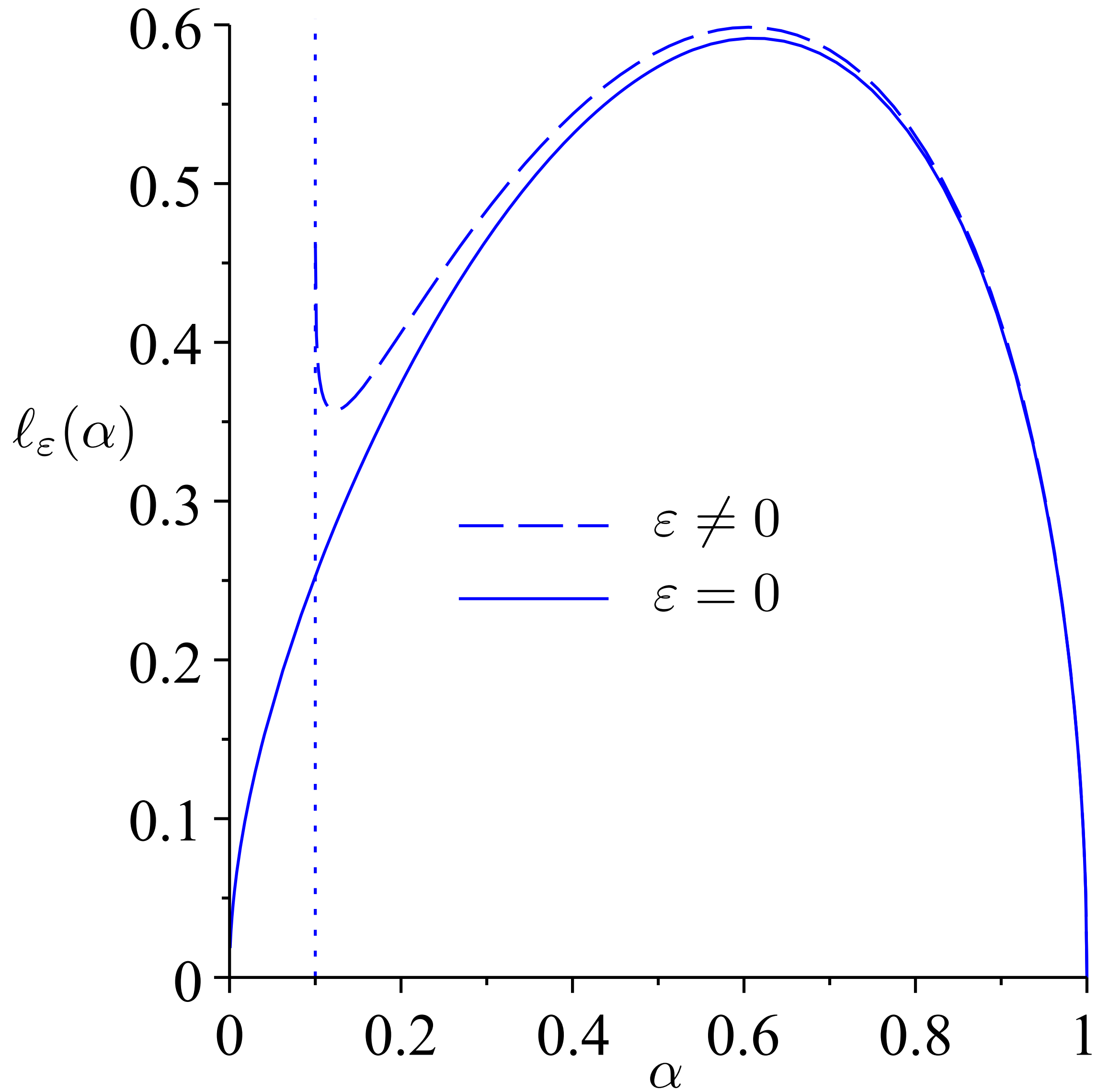
$$\begin{cases} u_y &= w \\ w_y &= \dfrac{1}{(1+u)^2} - \dfrac{\varepsilon^{m-2}}{(1+u)^m} \end{cases}$$

Saddle at $u = -1 + \varepsilon$



$$\ell_\varepsilon(\alpha) = \int_0^{\ell_\varepsilon} dy = \int_{-1+\alpha}^{0} \frac{du}{w} = \int_{-1+\alpha}^{0} \left[\frac{1}{\alpha} - \frac{1}{1+u} + \frac{\varepsilon^{m-2}}{m-1}\left(\frac{1}{(1+u)^{m-1}} - \frac{1}{\alpha^{m-1}}\right)\right]^{-\frac{1}{2}}$$

Equilibrium solutions correspond to roots of $\boxed{\ell_\varepsilon(\alpha) = \sqrt{\lambda}}$



Perturbation of Principal Fold (Pull-in - Voltage)

$$\begin{aligned} \lambda^*(\varepsilon) &= \lambda^*(0) + \varepsilon^{m-2}\lambda_1^* + \mathcal{O}(\varepsilon^{2(m-2)}) \\ &= 0.3500 + 0.7945\varepsilon^2 + \mathcal{O}(\varepsilon^4) \qquad (m=4) \end{aligned}$$

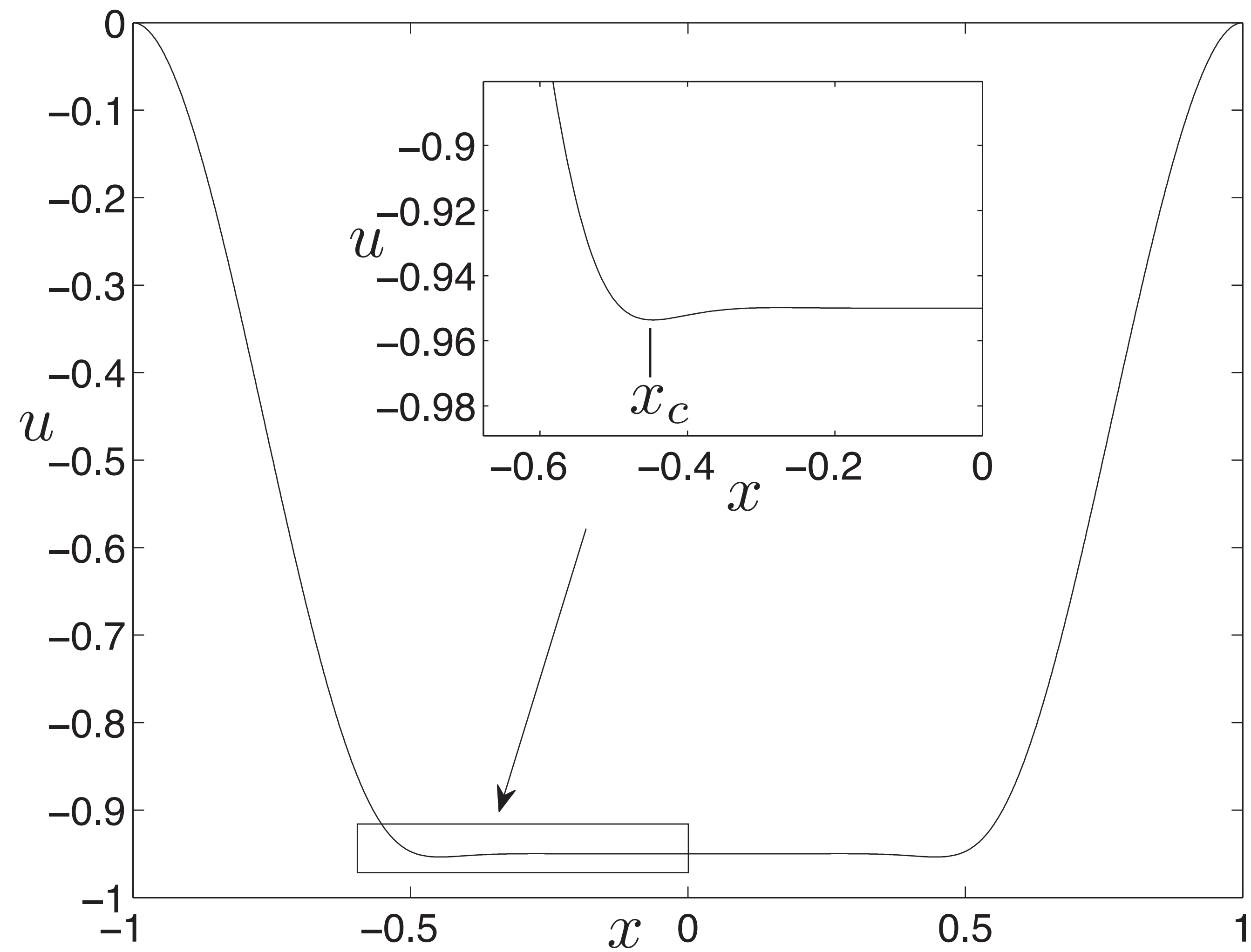
Existence of new solution branch:

$$-\frac{\varepsilon^{3/2}}{\sqrt{m-2}} \ln\left(\frac{\alpha}{\varepsilon} - 1\right) \lesssim l_\varepsilon(\alpha) \lesssim -\frac{\varepsilon^{1/2}}{\sqrt{2}} \sqrt{\frac{m-1}{m-2}} \ln\left(\frac{\alpha}{\varepsilon} - 1\right).$$

Equilibrium analysis in 4th order case.

Observations:

- Oscillatory Boundary layer implies single point contact.
- Large portion of beam in contact with substrate.
- Sharp boundary layer joining contact point with x = 1,-1.



$$\lambda = 25, \qquad \varepsilon = 0.005.$$

$$-u_{xxxx} = \frac{\lambda}{(1+u)^2} - \frac{\lambda \varepsilon^{m-2}}{(1+u)^m}, \quad x \in (-1,1); \qquad u = u_x = 0, \quad x = \pm 1$$

Goal: Calculate these post-contact states in the limit $\varepsilon \to 0$

Outline of Matched Asymptotic Analysis:

Step 1: Rescale: $x_c = 1 - \varepsilon^{\frac{1}{4}}\bar{x}_c, \qquad u(x) = w(\eta), \qquad \eta = \dfrac{x - (1 - \varepsilon^{\frac{1}{4}}\bar{x}_c)}{\varepsilon^{\frac{1}{4}}\bar{x}_c}$

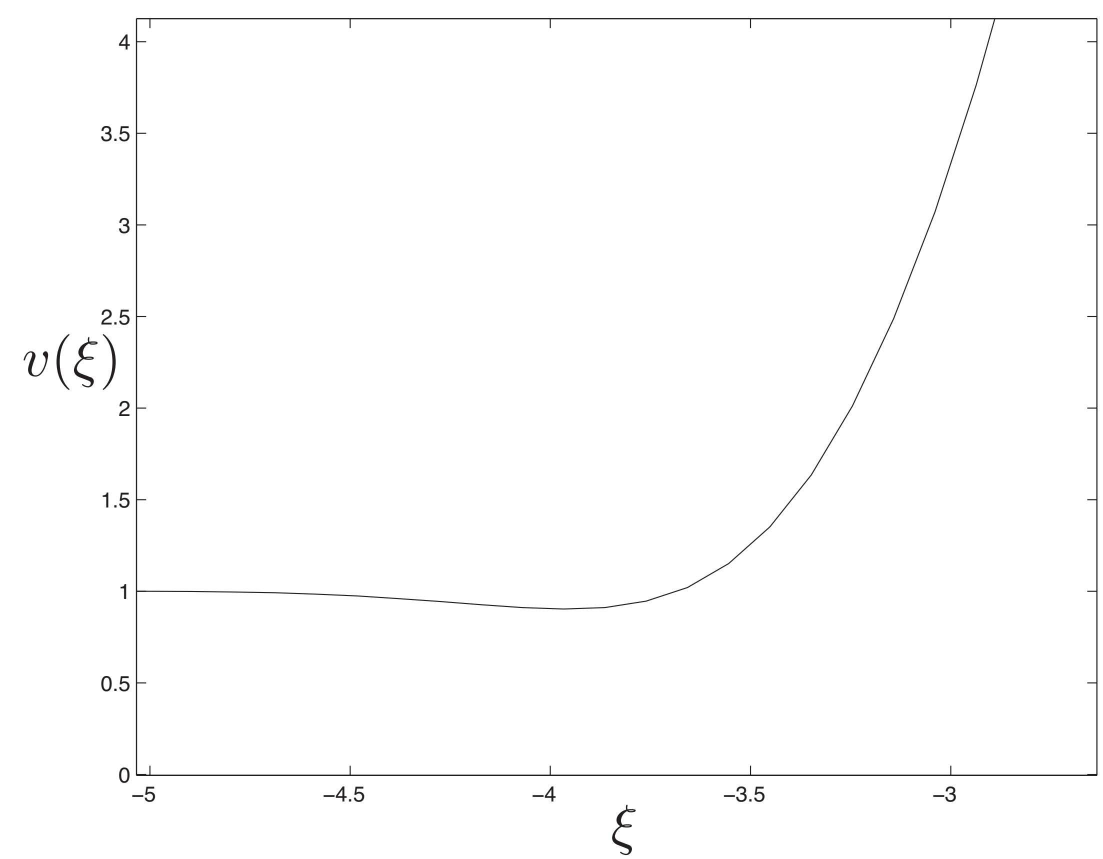
Step 2: Analyze contact layer: $w(\eta) = -1 + \varepsilon v(\xi), \qquad \xi = \varepsilon^q \eta.$

$$-v_{\xi\xi\xi\xi} = \lambda\left(\frac{1}{v^2} - \frac{1}{v^m}\right), \quad -\infty < \xi < \infty; \qquad v(0) = \arg\min_{\xi\in\mathbb{R}} v(\xi)$$

$$\lim_{\xi\to-\infty} v(\xi) = 1, \qquad -1 + \varepsilon v\left(\frac{\eta\,\bar{x}_c}{\varepsilon^{1/2}}\right) \sim w(\eta) \quad \text{as} \quad \xi = \frac{\eta\,\bar{x}_c}{\varepsilon^{1/2}} \to \infty.$$

Step 3: Expand solution

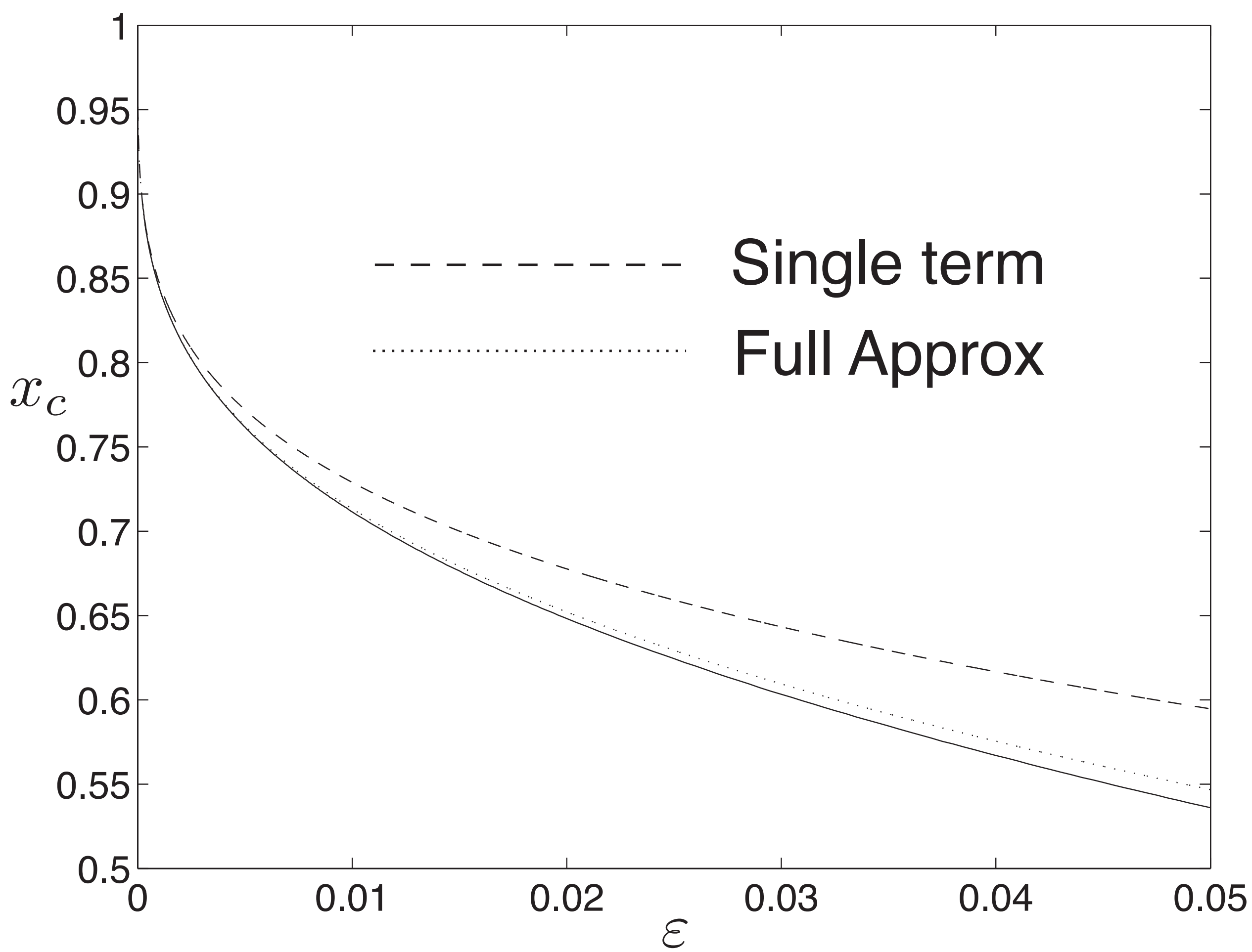
$$\begin{aligned} w &= w_0 + \varepsilon^{1/2} w_{1/4} + \varepsilon\log\varepsilon\, w_{1/2} + \mathcal{O}(\varepsilon); \\ \lambda_c &= \lambda_{0c} + \varepsilon^{1/2}\lambda_{1c} + \varepsilon\log\varepsilon\,\lambda_{2c} + \mathcal{O}(\varepsilon) \end{aligned}$$

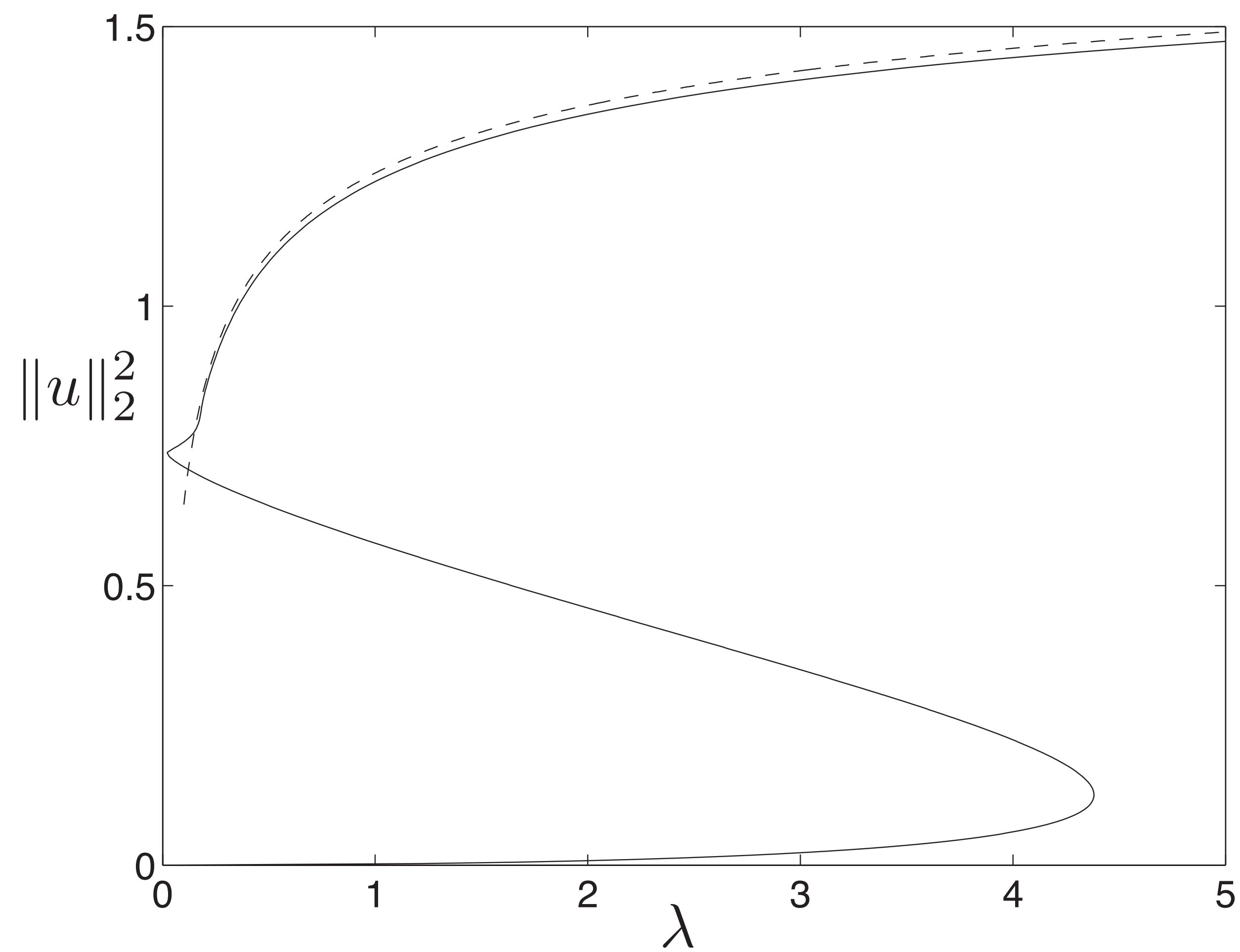


Results (after matching):

$$x_c = \pm\left[1 - \left[\frac{18(m-1)}{\lambda(m-2)}\right]^{1/4}\left(\varepsilon^{1/4} - \frac{\xi_0}{6}\varepsilon^{3/4} - \frac{\lambda_{0c}}{648}\varepsilon^{5/4}\log\varepsilon + \mathcal{O}(\varepsilon^{5/4})\right)\right].$$

$$\|u(x;\varepsilon)\|_2^2 = 2\left[1 - \frac{22}{35}\left(\frac{18(m-1)}{\lambda(m-2)}\right)^{1/4}\varepsilon^{1/4} + \mathcal{O}(\varepsilon^{3/4})\right].$$

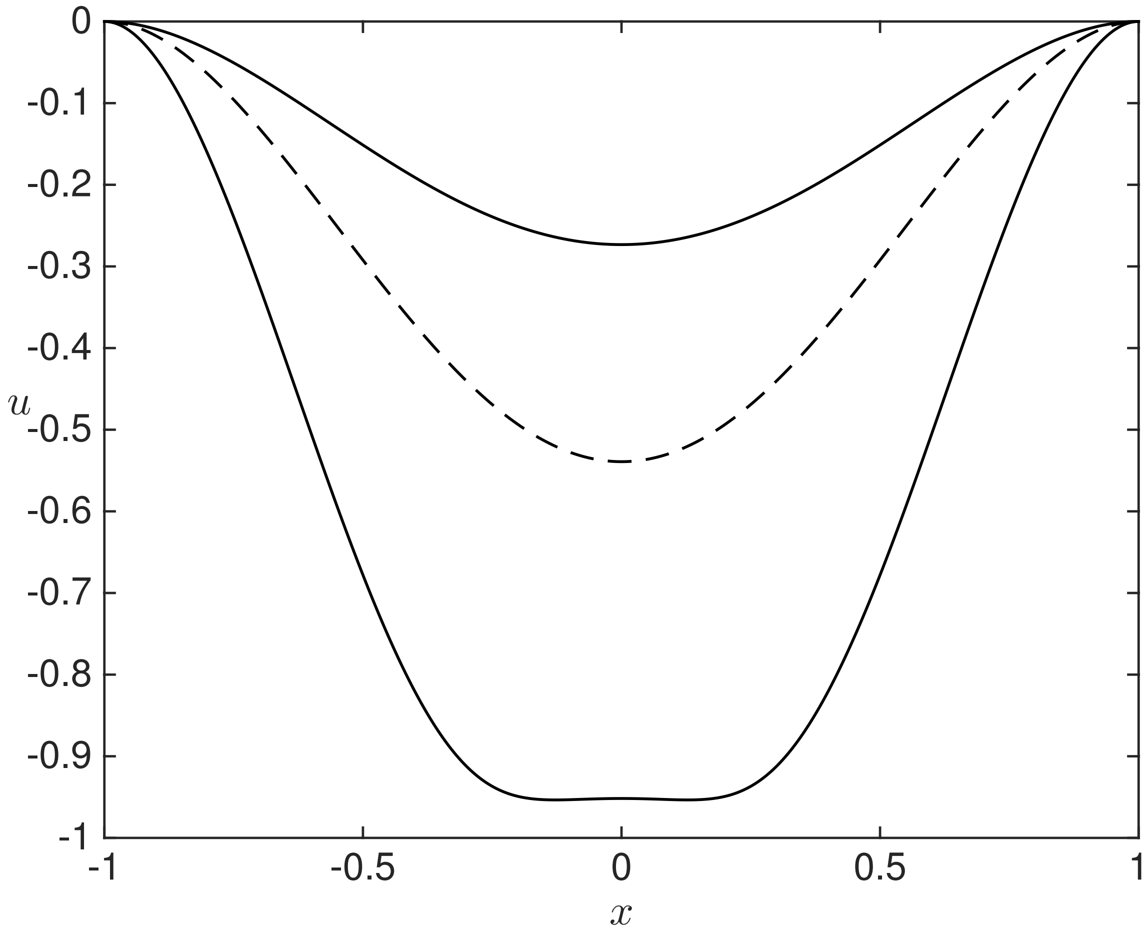




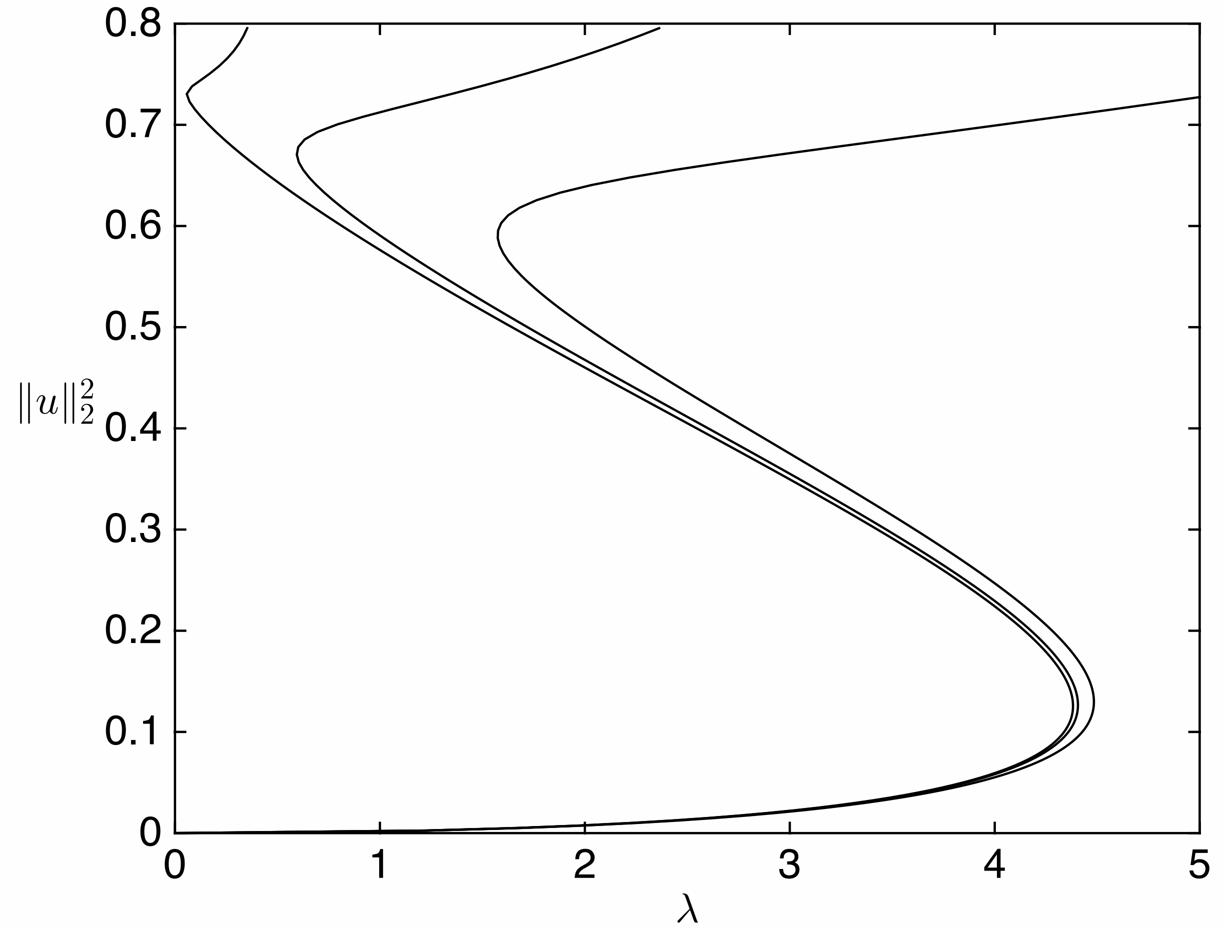
- In this regime, there is no approximation of the secondary fold point.
- This requires a separate singular analysis where $\lambda \to 0$

Fold Point Asymptotics

$$-u_{xxxx} = \frac{\lambda}{(1+u)^2} - \frac{\lambda \varepsilon^{m-2}}{(1+u)^m}, \quad x \in (-1,1); \qquad u = u_x = 0, \quad x = \pm 1$$



$\lambda = 4, \qquad \varepsilon = 0.05.$



$\varepsilon = 0.01, \; 0.05, \; 0.1.$

Local Parameterization of Bifurcation curve:

$$u(0) \quad = -1 + \varepsilon\alpha$$

$$\lambda \quad \sim \nu(\varepsilon)\lambda_0(\alpha), \qquad \nu \ll 1$$

Steps in the Analysis:

Step 1: Expand Outer Region (away from 0)

$$\begin{aligned} u &= u_0 + u_1\nu + \cdots \\ \lambda &= \lambda_0\nu + \cdots \end{aligned} \quad \Longrightarrow \quad \begin{gathered} u_0'''' = 0 \\ \boxed{u_0 = -1 + 3x^2 - 2x^3} \end{gathered}, \quad \begin{gathered} u_0(1) = 0, \quad u_0'(1) = 0 \\ u_0(0) = -1, \quad u_0'(0) = 0 \end{gathered}$$

Step 2: Blowup Inner Region (near to 0)

$$\begin{aligned} u(x) &= -1 + \varepsilon w(y) \\ x &= \varepsilon^{\frac{1}{2}} y \end{aligned} \quad \Longrightarrow \quad \begin{gathered} \boxed{\nu(\varepsilon) = \varepsilon^{\frac{3}{2}}} \\ w \sim 3y^2 - 2\varepsilon^{\frac{1}{2}} y^3 + \cdots \quad y \to \infty \end{gathered}$$

$$-w'''' = \lambda_0 \varepsilon^{\frac{1}{2}} \left[\frac{1}{w^2} - \frac{1}{w^m} \right], \quad y > 0; \qquad \begin{gathered} w(0) = \alpha, \\ w'(0) = w'''(0) = 0 \end{gathered}$$

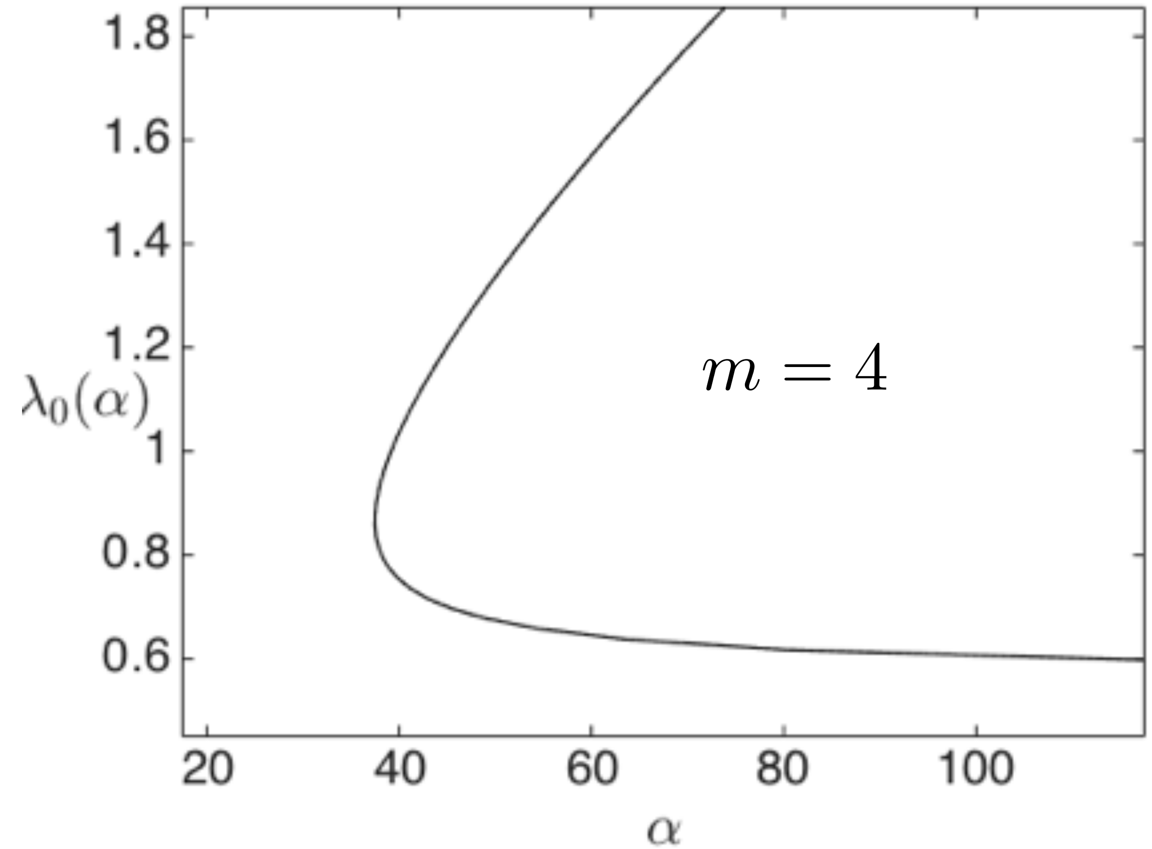
Step 3: Match

The value of $\lambda_0(\alpha)$ which gives the correct growth as $y \to \infty$:

$$\boxed{\lambda_0(\alpha) = 12\sqrt{3} \left[\frac{\pi}{4\alpha^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{2\alpha^{m-\frac{1}{2}}} \frac{\Gamma(m - \frac{1}{2})}{\Gamma(m)} \right]^{-1}, \qquad m > 2.}$$

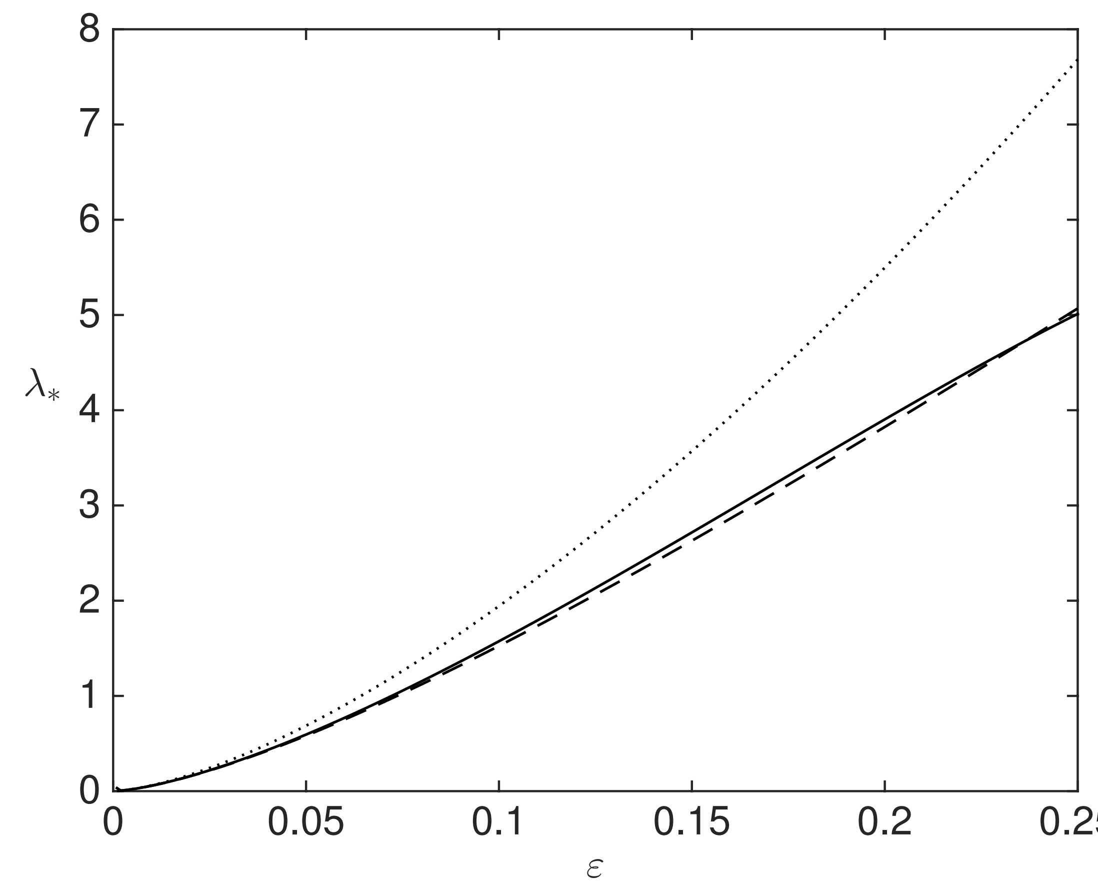
Results

$$\lambda_0(\alpha) = 12\sqrt{3}\left[\frac{\pi}{4\alpha^{\frac{3}{2}}} - \frac{\sqrt{\pi}}{2\alpha^{m-\frac{1}{2}}}\frac{\Gamma(m-\frac{1}{2})}{\Gamma(m)}\right]^{-1}$$



After a lot of algebra the second term is forthcoming

$$\begin{aligned}\lambda_* \quad &\sim \quad \frac{7}{\pi}\left(\frac{105^3}{2}\right)^{\frac{1}{4}}\varepsilon^{\frac{3}{2}} - \frac{26411}{64\pi^2}\varepsilon^2 + \cdots \\ &\sim \quad 61.4586\,\varepsilon^{\frac{3}{2}} - 41.8124\,\varepsilon^2 + \cdots .\end{aligned}$$



Prediction of Cubic Fold Point

Primary Fold Asymptotics:

$$\begin{aligned}\lambda^*(\varepsilon) &= \lambda_0 + \varepsilon^{2(m-2)}\lambda_1 + \mathcal{O}(\varepsilon^{4(m-2)}),\\ &= 4.3809 + 9.9713\,\varepsilon^2 + \mathcal{O}(\varepsilon^4) \qquad [m=4]\end{aligned}$$

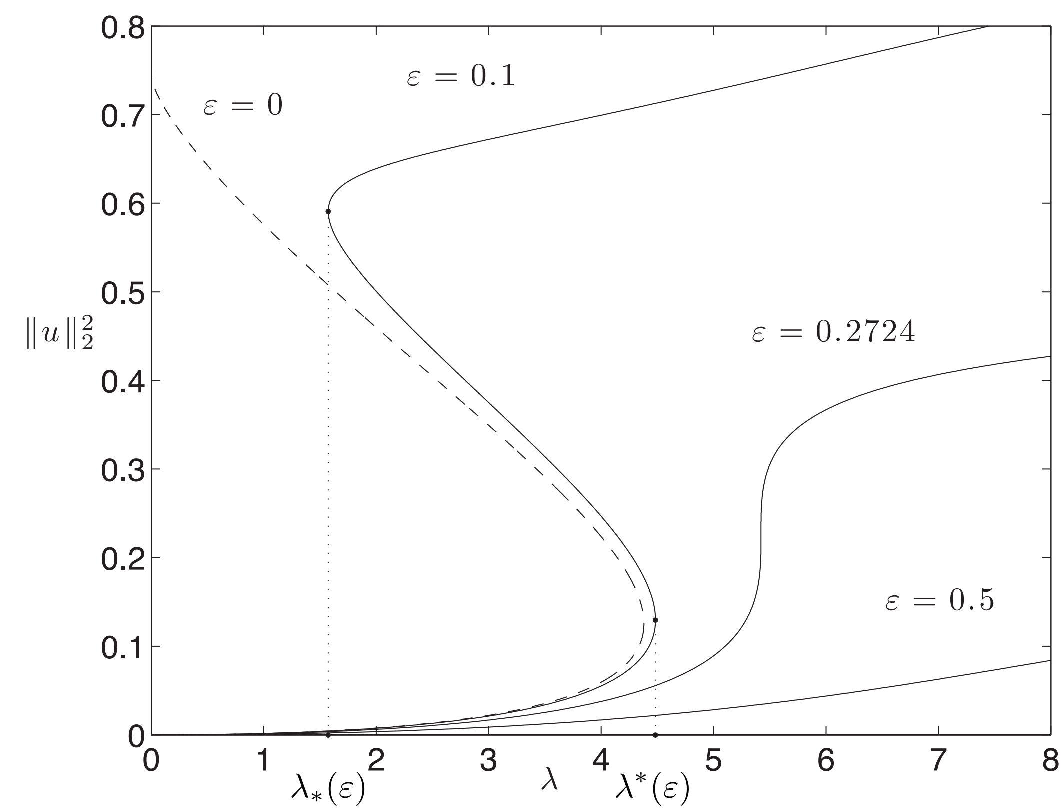
Bistable Range:

$$\lambda_*(\varepsilon) < \lambda < \lambda^*(\varepsilon)$$

$$61.4586\,\varepsilon^{\frac{3}{2}} - 41.8124\,\varepsilon^2 < \ \lambda < 4.3809 + 9.9713\,\varepsilon^2.$$

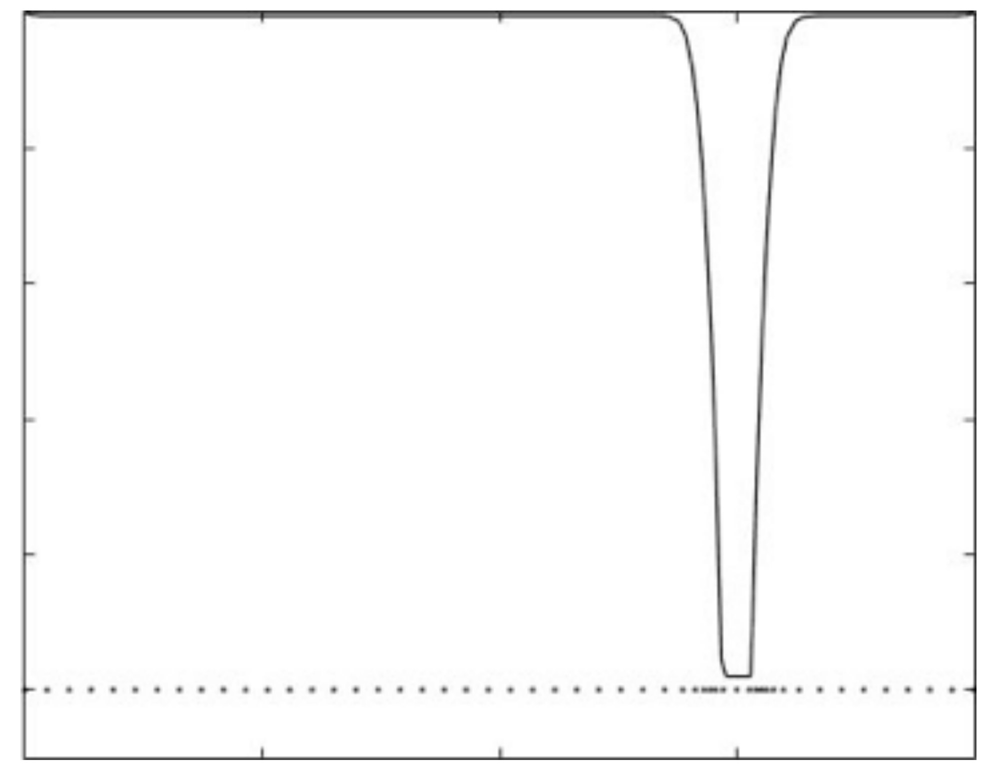
Critical value (m=4):

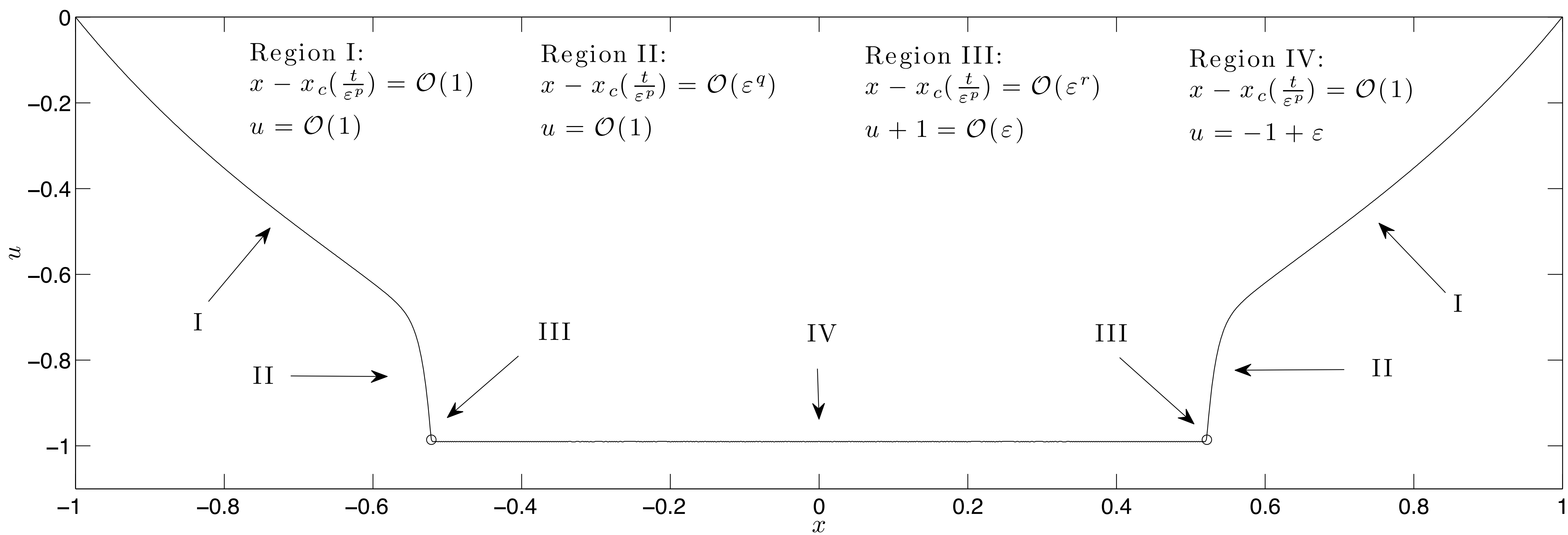
$$\boxed{\lambda_*(\varepsilon_c) = \lambda^*(\varepsilon_c) \quad \Longrightarrow \quad \varepsilon_c = 0.2468}$$



Sharp Interface limit dynamics

- Triple Deck Problem: Boundary Layer inside boundary layer.
- Notorious in high-Reynolds number flows.





Outline of steps

Innermost Layer:

$$y = \frac{x - x_c(\frac{t}{\varepsilon^{1/2}})}{\varepsilon^{3/2}}, \qquad u = -1 + \varepsilon\, v(y,t)$$

$$v_{0yy} = \frac{\lambda}{v_0^2} - \frac{\lambda}{v_0^m}, \quad y \in \mathbb{R}; \qquad v_0(0) = \left(\frac{m}{2}\right)^{\dfrac{1}{m-2}},$$

$$v_0 = 1 + \cdots \quad y \to -\infty; \qquad v_0 = \sqrt{\frac{2\lambda(m-2)}{m-1}}\, y + \cdots,$$

Intermediate Layer:

$$z = \frac{x - x_c(\frac{t}{\varepsilon^{1/2}})}{\varepsilon^{1/2}}, \qquad u(x,t) = \varepsilon\, v(z,t)$$

$$w_{zz} + \dot{x}_c\, w_z = 0, \quad z > 0; \qquad w(0) = -1, \quad w_z(0) = \alpha;$$

$$w(z) = -1 + \frac{\alpha}{\dot{x}_c}\left[1 - e^{-\dot{x}_c\, z}\right] \sim -1 + \frac{\alpha}{\dot{x}_c}.$$

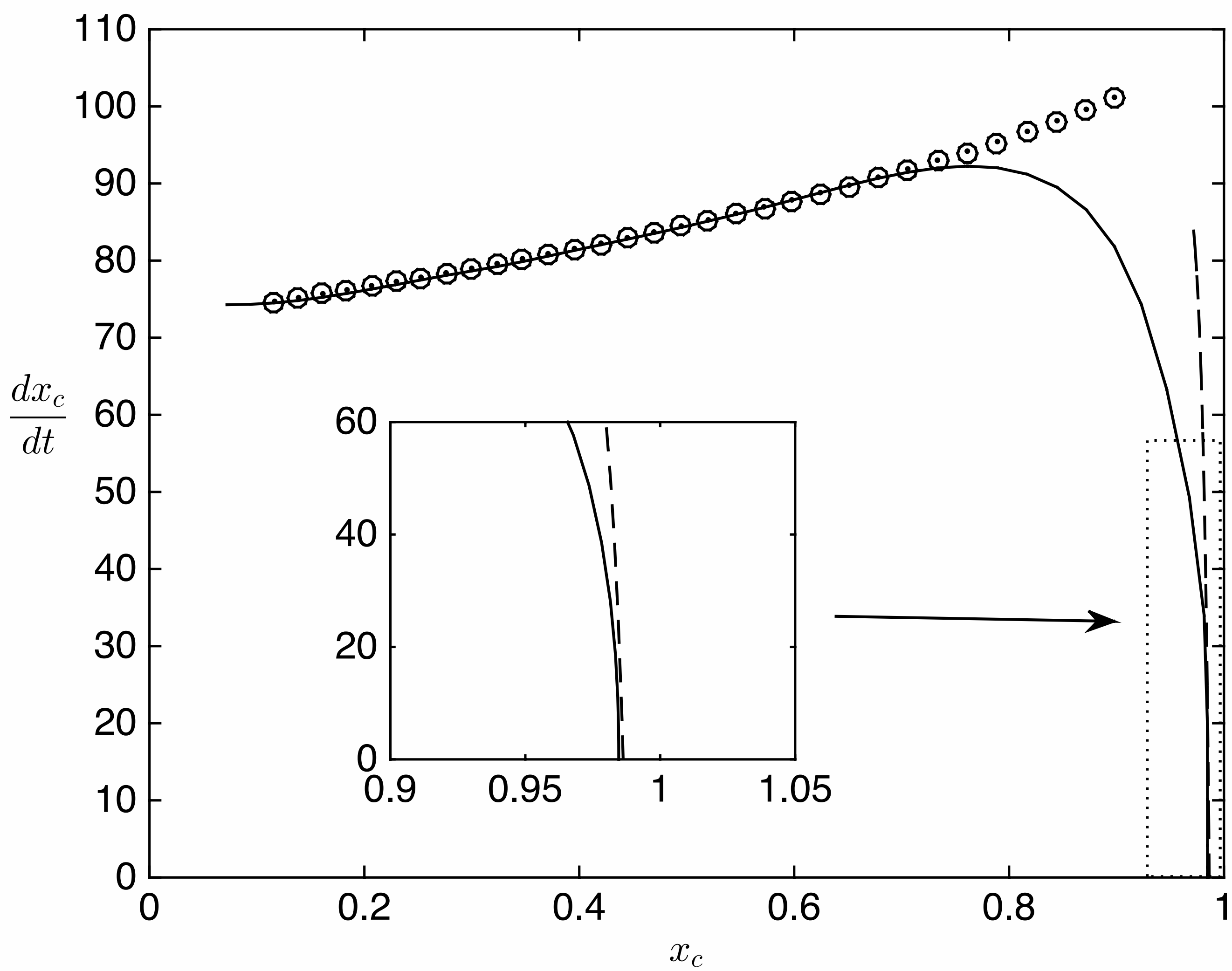
Outermost layer:

$$u(x_c,t) \sim -1 + \frac{\alpha}{\dot{x}_c}, \quad \Longrightarrow \quad \boxed{\frac{dx_c}{dt} \sim \frac{\alpha}{\varepsilon^{1/2}(1 + u(x_c,t))}, \quad \alpha = \sqrt{\frac{2\lambda(m-2)}{m-1}}}$$

1D Interface Dynamics

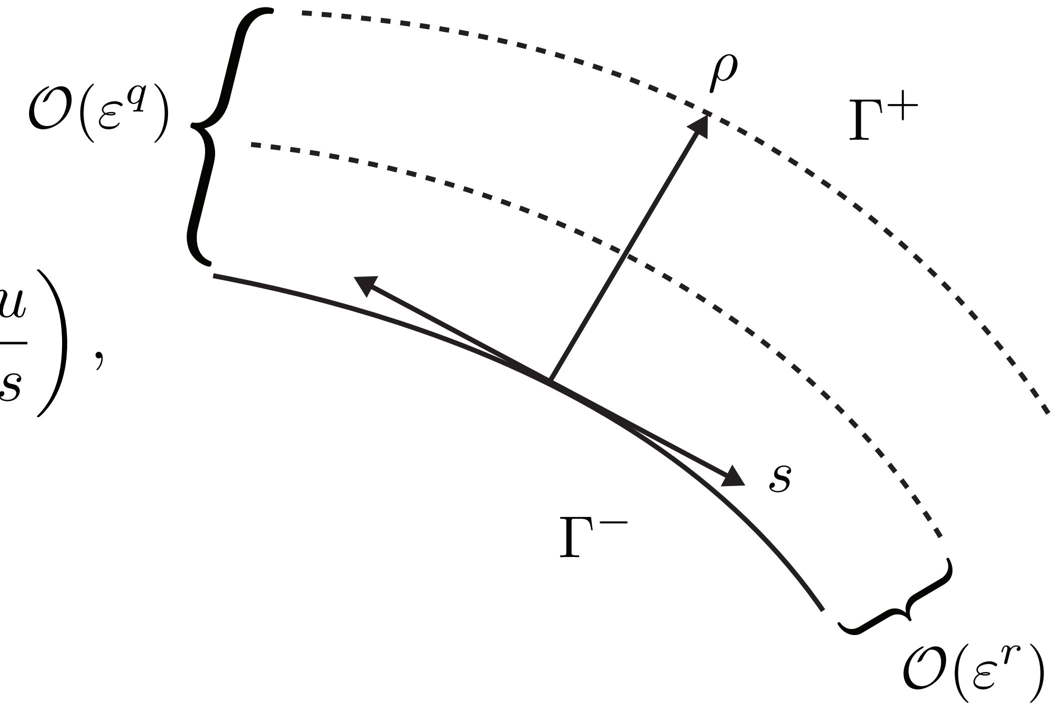
The full three term expansion!

$$\frac{dx_c}{dt} \sim \frac{\alpha}{\varepsilon^{\frac{1}{2}}[1+u_0(x_c)]} - \frac{\lambda}{\alpha}\frac{\varepsilon^{\frac{1}{2}}\log\varepsilon}{[1+u_0(x_c)]^2} + \frac{\varepsilon^{\frac{1}{2}}}{[1+u_0(x_c)]^2}\left[\alpha a_1 - \frac{3\lambda}{\alpha} - \frac{\lambda}{\alpha}\log\frac{\alpha}{[1+u_0(x_c)]}\right].$$



Separate analysis when the layer meets the boundary and decelerates.

Interface dynamics in 2D



$$
\begin{aligned}
\Delta u \quad &\equiv \frac{\partial^2 u}{\partial \rho^2} - \frac{\kappa(s)}{1-\kappa(s)\rho}\frac{\partial u}{\partial \rho} + \frac{1}{1-\kappa(s)\rho}\frac{\partial}{\partial s}\left(\frac{1}{1-\kappa(s)\rho}\frac{\partial u}{\partial s}\right),\\
\frac{du}{dt} \quad &\equiv \frac{\partial u}{\partial t} + \rho_t\frac{\partial u}{\partial \rho} + s_t\frac{\partial u}{\partial s},
\end{aligned}
$$

Interface normal velocity: Laplacian case

$$
\boxed{\rho_t = \frac{\alpha}{\varepsilon^{1/2}(1+u_0(x_c))} - \kappa + \cdots\,, \qquad \alpha = \sqrt{\frac{2\lambda(m-2)}{m-1}}}
$$

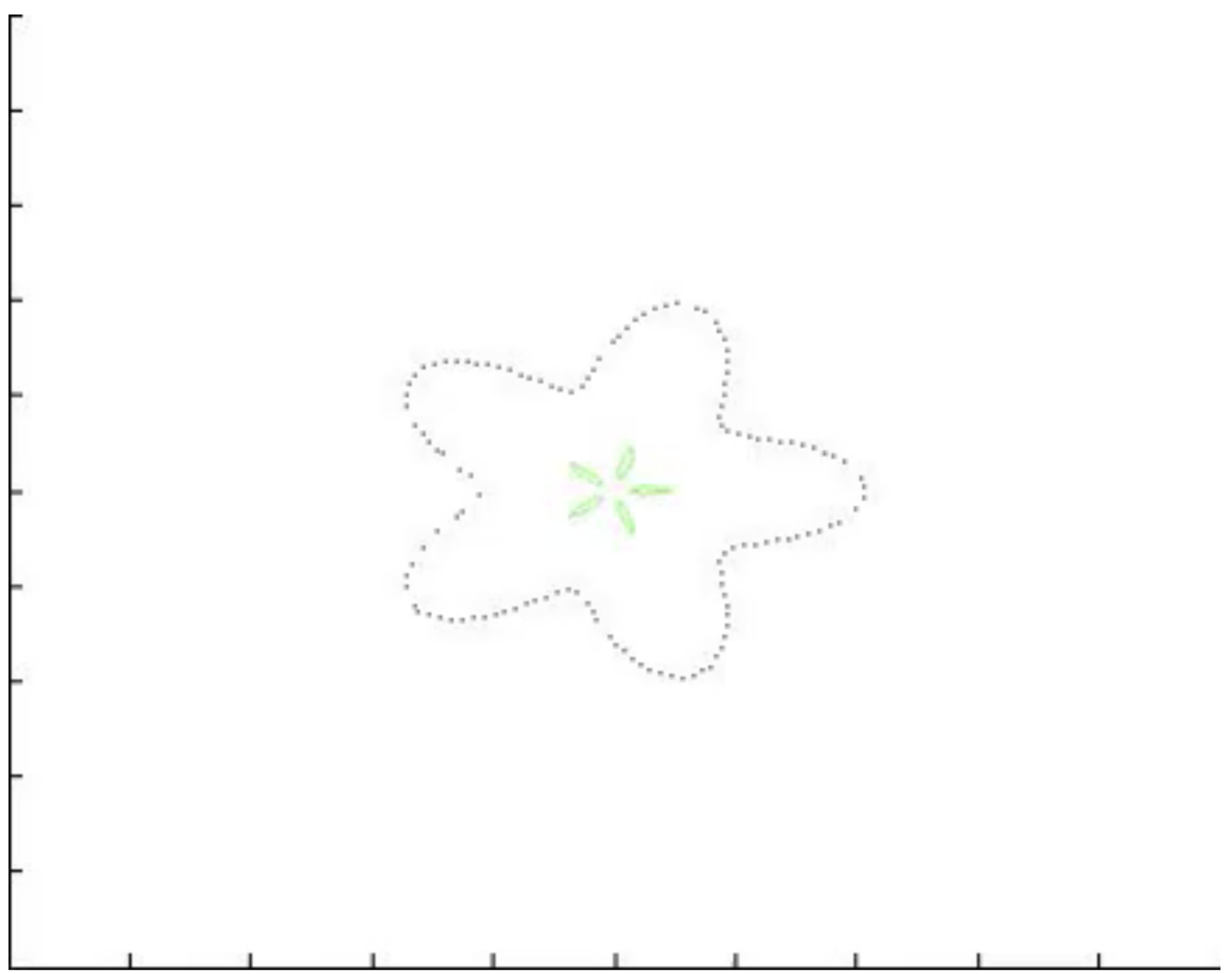
Interface normal velocity: Bi-Laplacian case

$$
\boxed{\rho_t \sim \left[\frac{2\alpha}{\varepsilon^{\frac{1}{2}}(1+u_0)}\right]^{3/2} - \frac{2\alpha}{\varepsilon^{\frac{1}{2}}(1+u_0)}\kappa + \cdots\,, \qquad \alpha = \sqrt{\frac{\lambda(m-2)}{2(m-1)}}}
$$

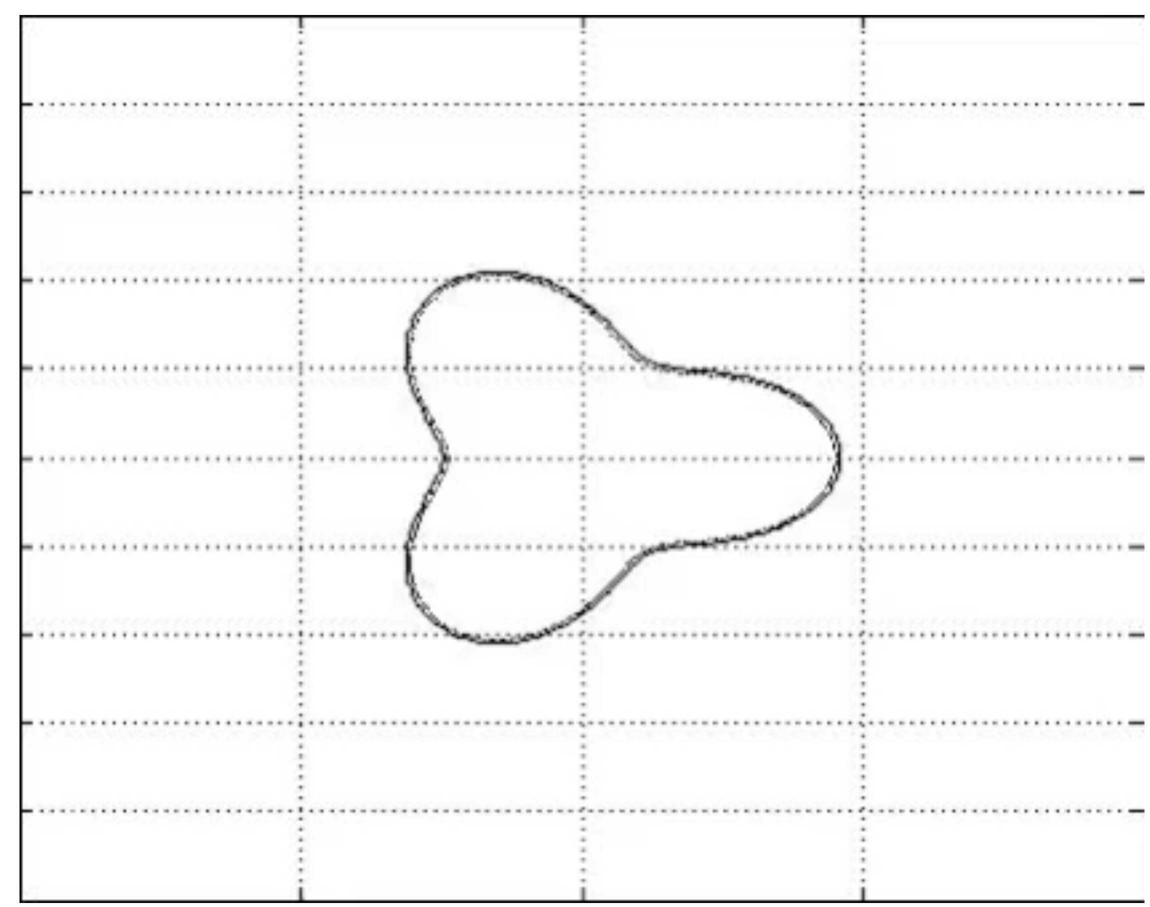
Results - interface law evolved by level set method.

$$\phi(x(t), y(t), t) = 0 \qquad \Longrightarrow \qquad \phi_t + \rho_t |\nabla \phi| = 0$$

$$\rho_t = \gamma_1(t) + \gamma_2(t)\kappa \qquad \kappa = \frac{\phi_{xx}\phi_y^2 - 2\phi_{xy}\phi_x\phi_y + \phi_{yy}\phi_x^2}{\phi_x^2 + \phi_y^2}$$



Laplacian $\lambda = 10, \quad \varepsilon = 0.05.$



Bi-Laplacian: $\lambda = 2000, \quad \varepsilon = 0.005.$

Summary

- Blow up in fourth order PDEs are extremely sensitive to parameters/geometry.
- Regularization gives rise to new singular stable solutions.
- Characterization of new stable equilibrium in 1D and dynamics.
- Stiff numerical problems require careful numerics to adapt to solution features.

Thank you for your attention!

References (available at: nd.edu/~alindsa1)

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