Problems in diffusion and absorption: How fast can you hit a target with a random walk?

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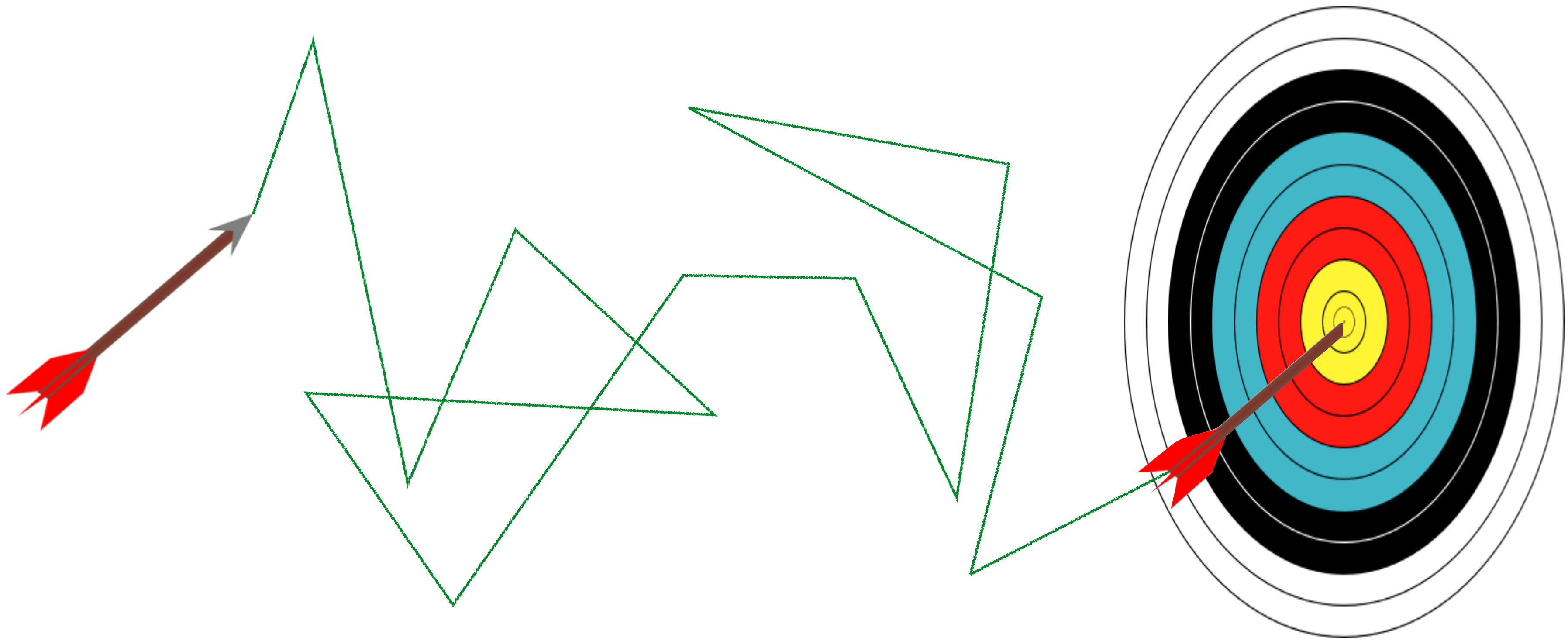
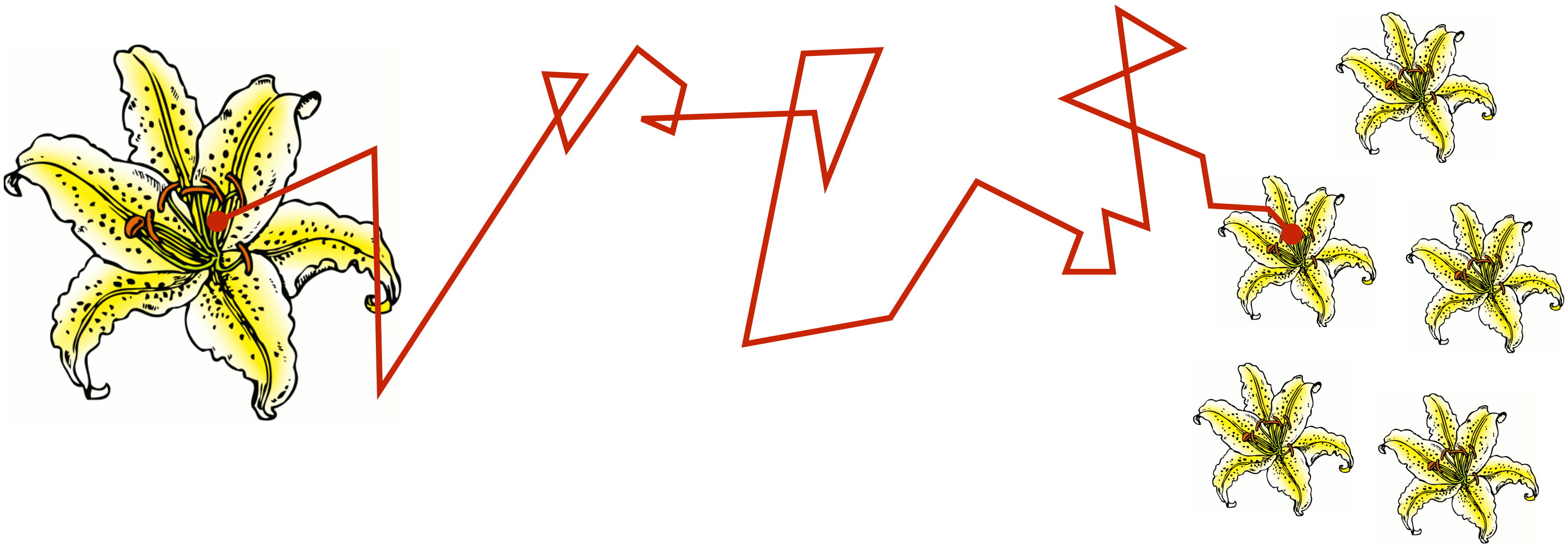
Random Signaling Problems

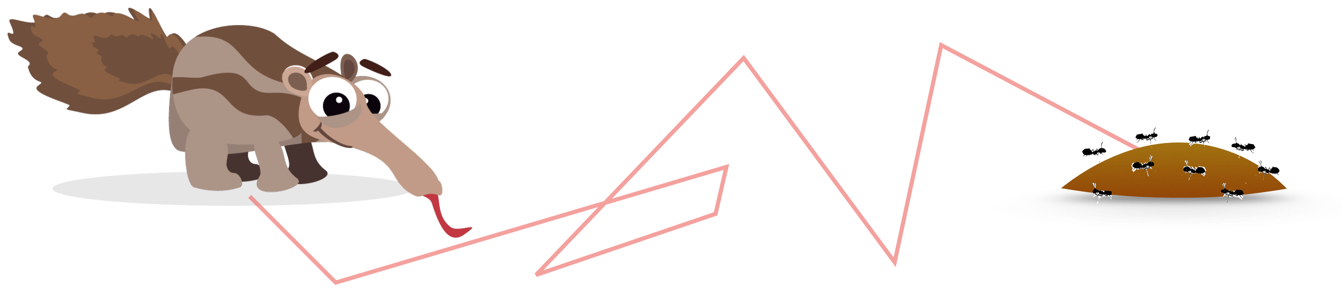


Pollination



- *What what probability will a pollen grain released from the stamen of one flower find the pistil of another flower?*
- *On average, how long does this take?*
- *How does the shape of the stamen affect this?*

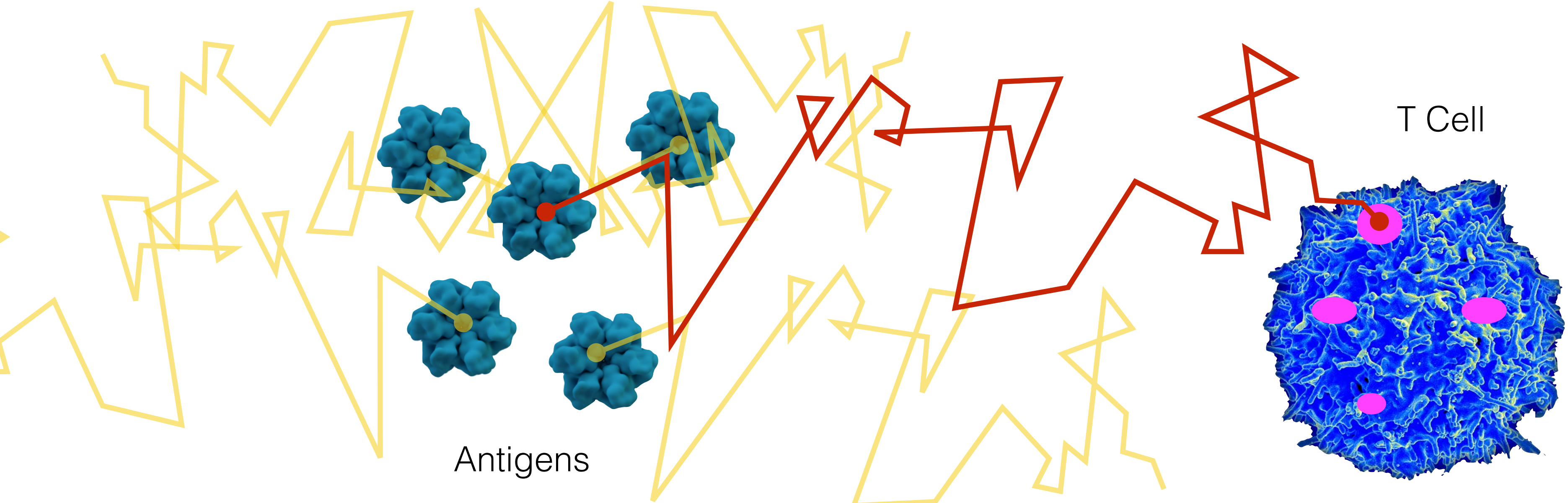
Foraging



Suppose an animal is randomly foraging for food.

- *Will the anteater find the ants?*
- *On average, how long does this take?*

Molecular Signaling



When an antigen (a toxin or a protein that promotes an immune response) binds to a receptor on a T-cell it can trigger the creation of antibodies.

- *What is the probability of this binding occurring?*
- *On average, how long does this take?*
- *How does the distribution of the receptors affect this?*

Modeling Diffusion
&
Capture

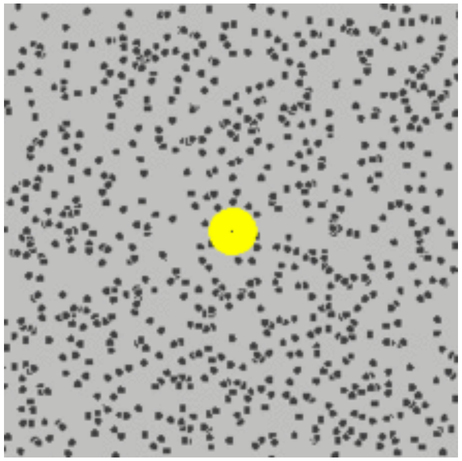
pollen on a stamen

http://www.dailymail.co.uk/sciencetech/article-1267236/Microscopic-images-reveal-invisible-enemy-faced-hayfever-sufferers.html

Brownian Motion

Brownian motion is the random walk a molecule takes due to collisions with other molecules.

Typically it can be describe by a distribution of velocities.



https://en.wikipedia.org/wiki/Brownian_motion

Let

- V_0 be the average speed
- L_0 be the mean free path

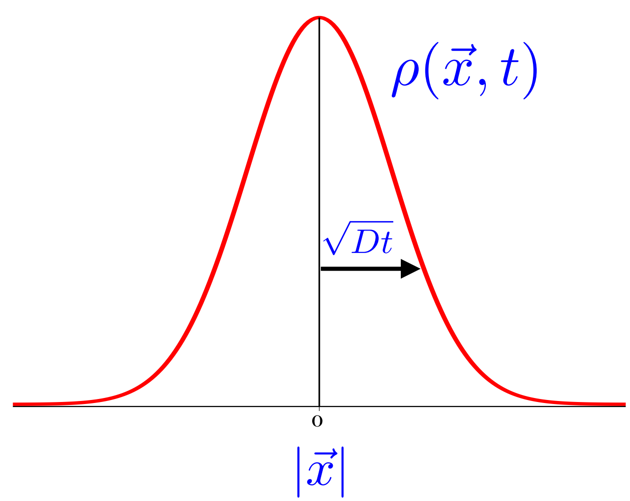
Random Walks

Due to the **Central Limit Theorem** after many steps the density, $\rho(\vec{x},t)$, of the random walk converges to a Gaussian distribution.

In N-dimensions, the distribution approaches

$$\rho(\vec{x},t) = \left(\frac{1}{\sqrt{4\pi D t}}\right)^N e^{-|\vec{x}|^2/(4Dt)}$$

$$\vec{x} \in \mathbb{R}^N$$



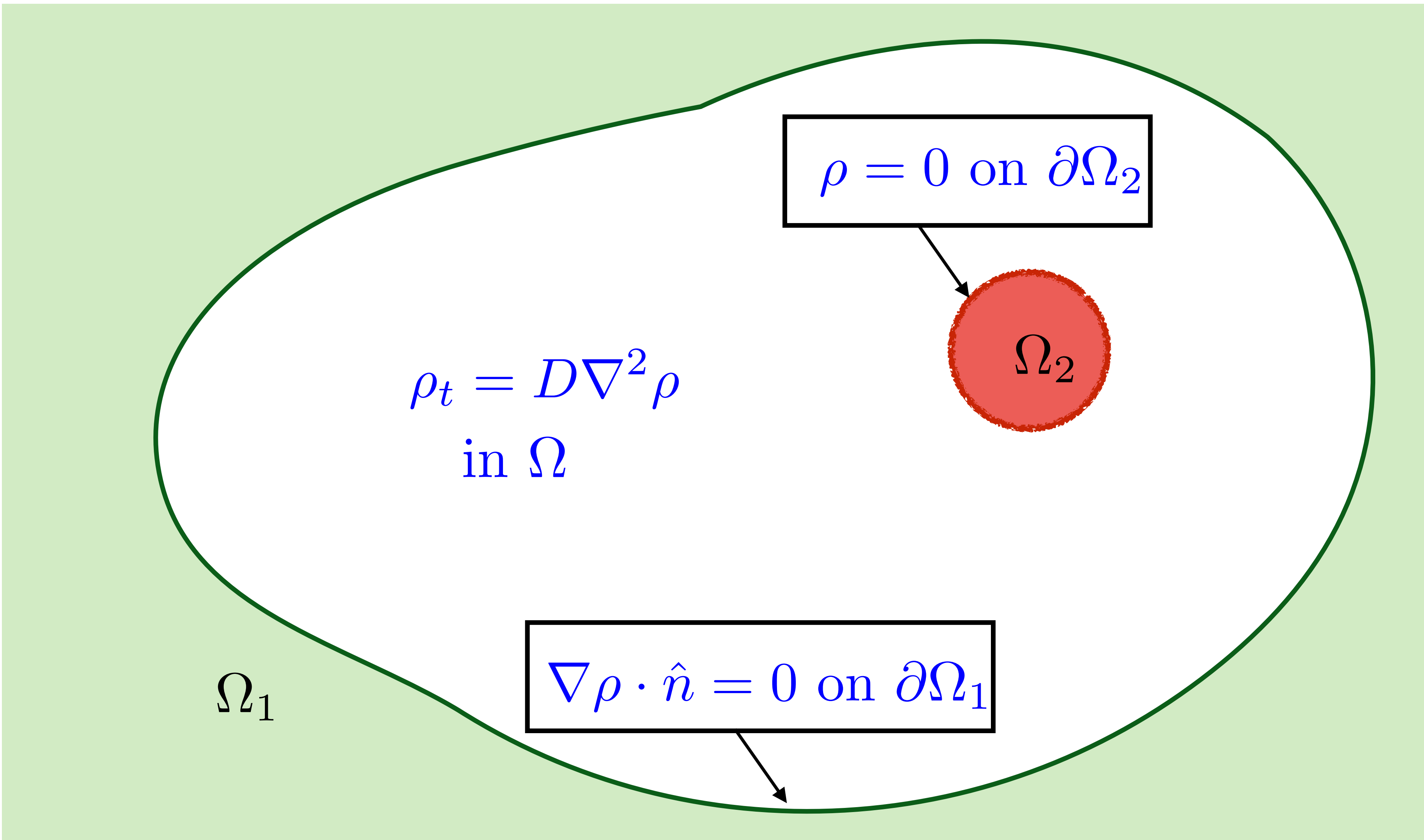
The diffusion constant, D, scales as

$$D \propto (V_0)^2/L_0$$ [Length]^2 x [Time]

We call $\rho(\vec{x},t)$ the *diffusion kernel.*

The Diffusion Equation

The density of molecules, $\rho(\vec{x}, t)$, satisfies the diffusion equation in some domain Ω .



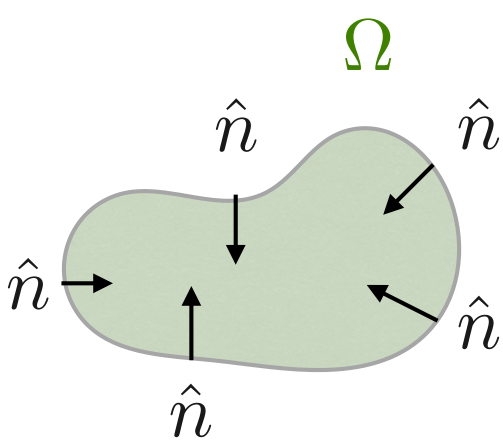
Flux & Neumann BC's

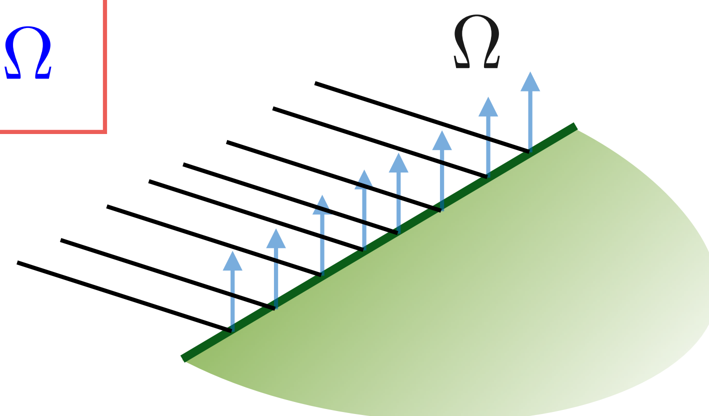
The flux of molecules is given by the Fourier Law

$$\vec{F} = D\nabla u \qquad \text{[velocity]x[density]}$$

At an impermeable boundary, we have a no-flux BC:

$$\vec{F} \cdot \hat{n} = 0 \quad \Rightarrow \quad \boxed{\nabla u \cdot \hat{n} = 0 \text{ on } \partial\Omega}$$



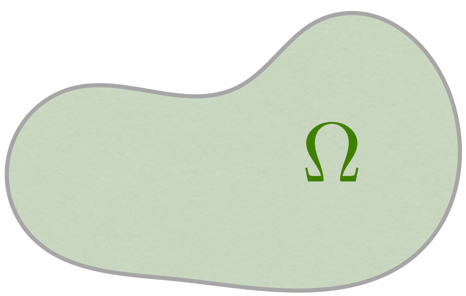


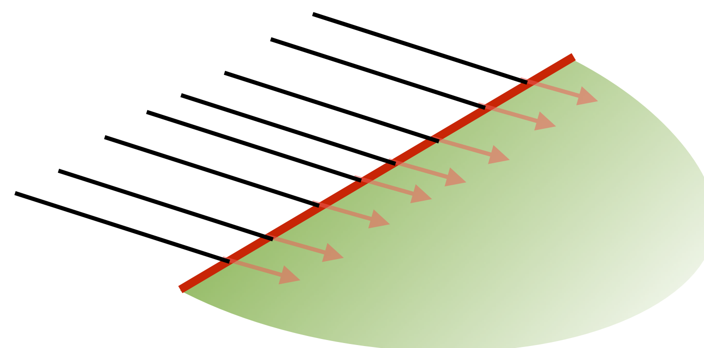
At the particle level, particles are reflected.
Angle of incidence =Angle of reflection

Capture & Dirichlet BC's

At a boundary that captures every molecule that impacts it the density will drop to zero

$$\boxed{u = 0 \text{ on } \partial\Omega}$$





At the particle level, particles are captured.

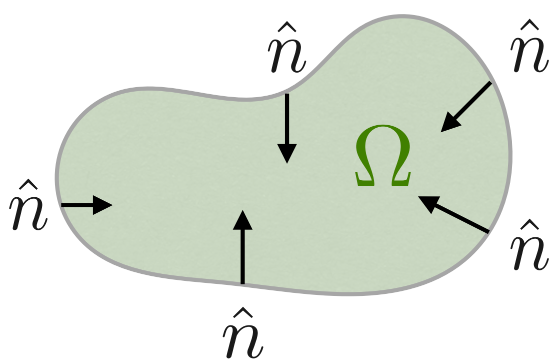
Partial Capture & Robin BC's

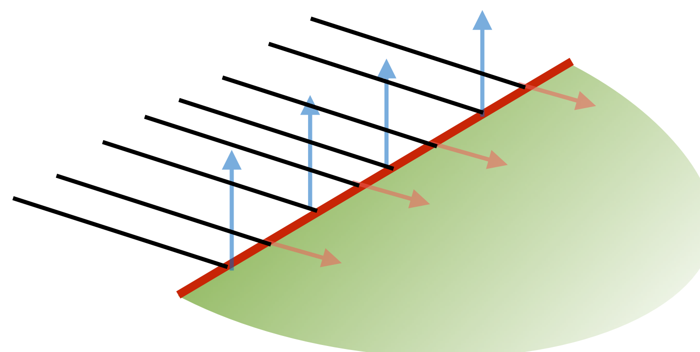
Consider a boundary that reflects some, but not all, of the particles that strike it.

$$\boxed{\nabla u \cdot \hat{n} + \kappa u = 0 \text{ on } \partial\Omega}$$

$\kappa \ll 1$ Mostly Reflecting

$\kappa \gg 1$ Mostly Absorbing





At the particle level, some particles are captured while some are reflected.

Metrics for Capture

Probability of Capture

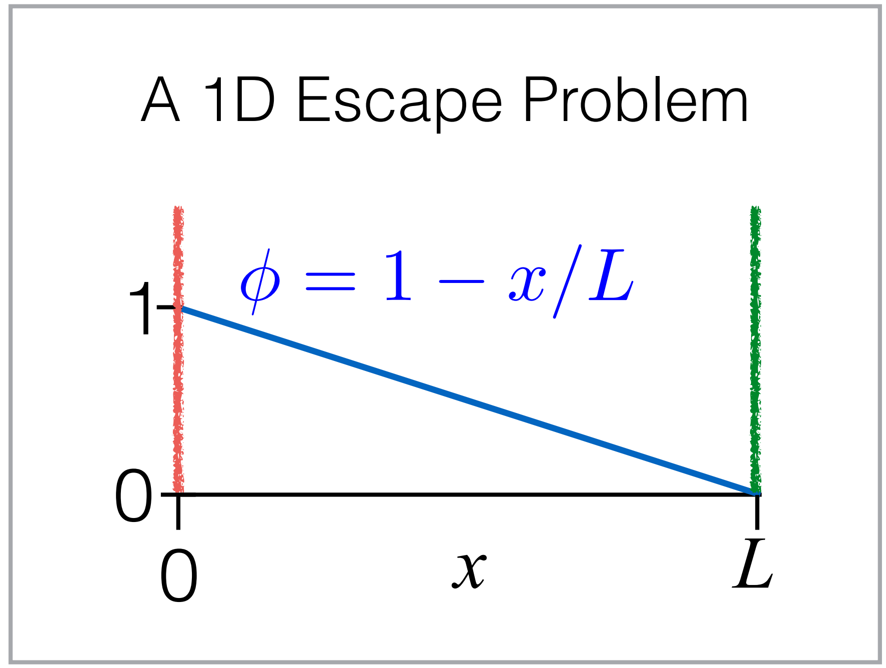
Consider a domain with a target boundary and an escape boundary.

The probability, $\phi(x)$, that a molecule is captured satisfies Laplace's equation:

$$\nabla^2\phi = 0 \text{ in } \Omega$$

$$\phi = 1 \quad \text{on } \partial\Omega \text{ for capture}$$

$$\phi = 0 \quad \text{on } \partial\Omega \text{ for escape}$$



This is sometimes called *harmonic measure*.

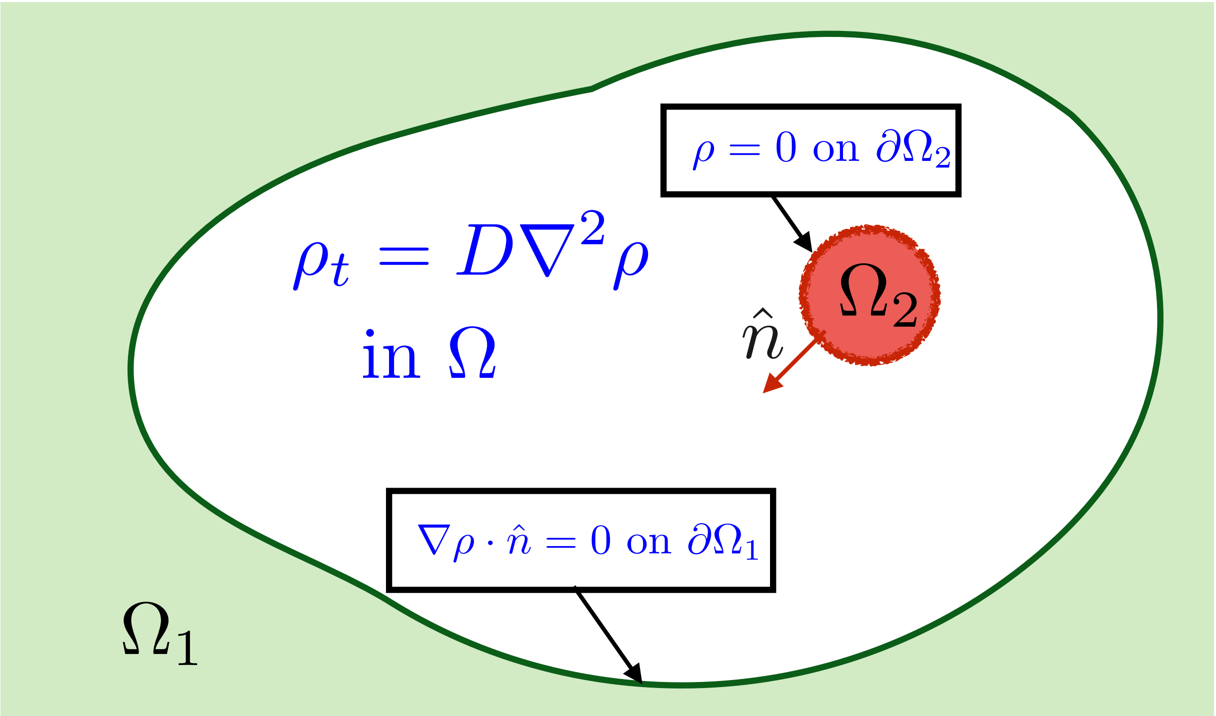
Distribution of First Passage Time (FPT)

Consider a domain with a target boundary.

The density of mass capture per unit time, *p(t)*, can be computed by integrating the flux over the target boundary.

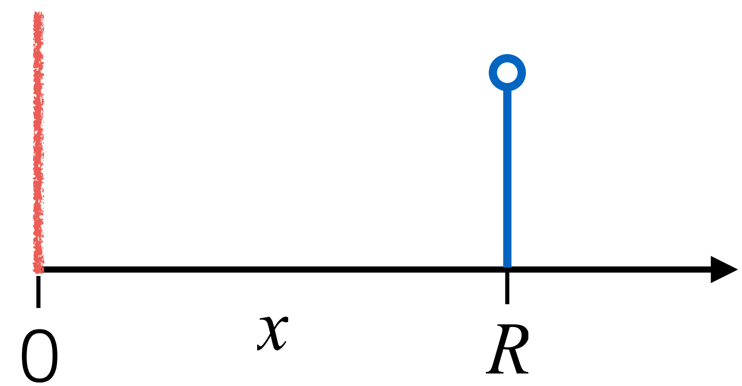
$$p(t) = -\oint_{\Omega_2} D\nabla\rho \cdot \hat{n}\, dS$$

$$\frac{[\text{Mass}]}{[\text{Time}]}$$

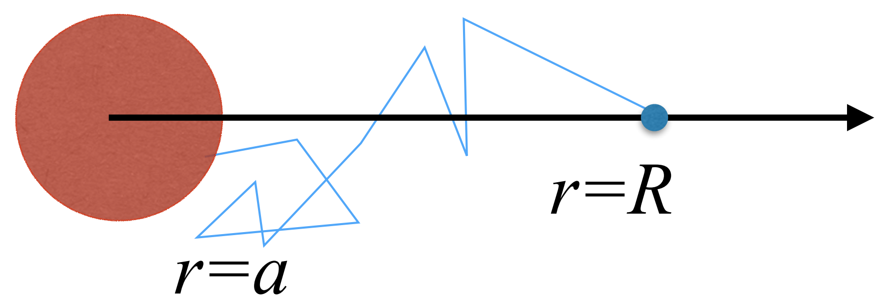


The Importance of Dimension

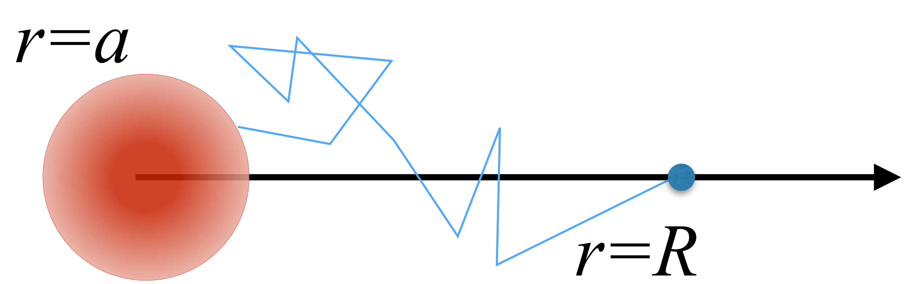
1D



2D

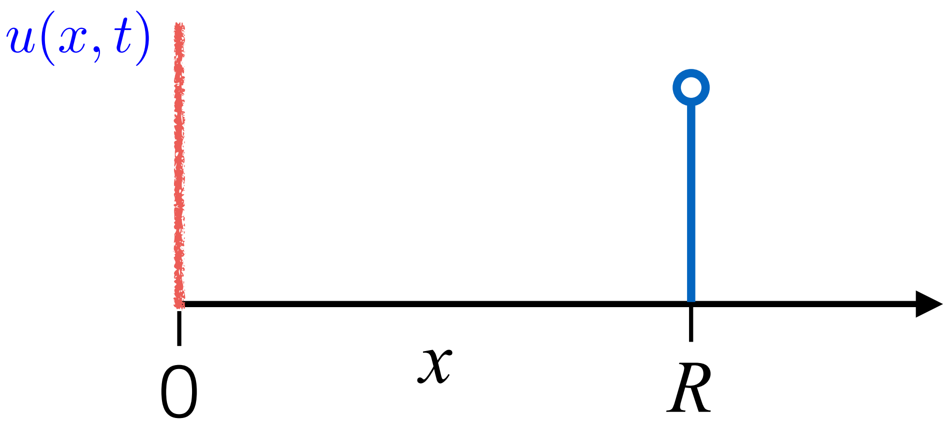


3D



Capture in 1D

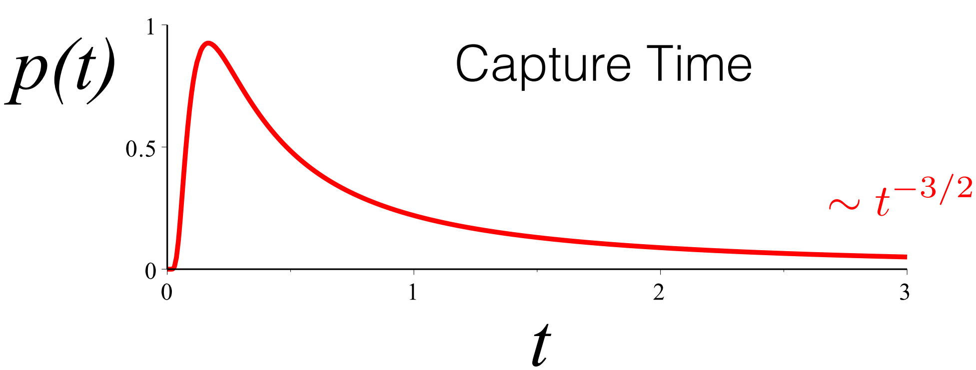
Consider a δ-function release at $x=R$.



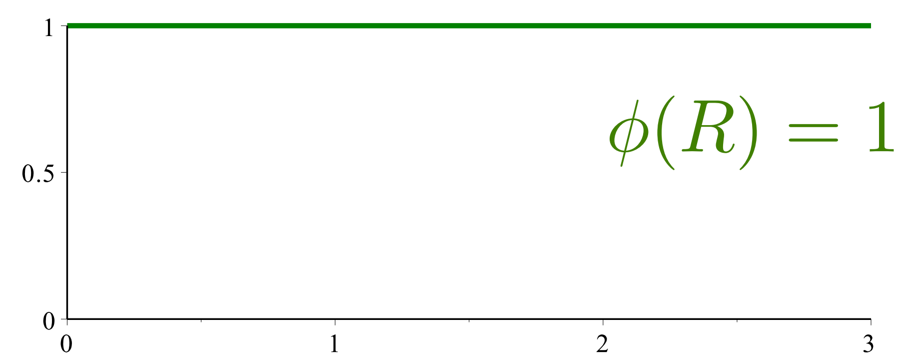
$$u_t = D u_{xx} \qquad x > 0,\ t > 0$$
$$u(x,0) = \delta(x - R) \qquad x > 0$$
$$u(0,t) = 0 \qquad t > 0$$

By method of images: $u(x,t) = \dfrac{1}{2\sqrt{\pi D t}}[e^{-(X-R)^2/4Dt} - e^{-(X+R)^2/4Dt}]$

FPT: $p(t) = -D u_x(0,t) = \dfrac{R e^{-R^2/4Dt}}{2\sqrt{\pi D} t^{3/2}}$

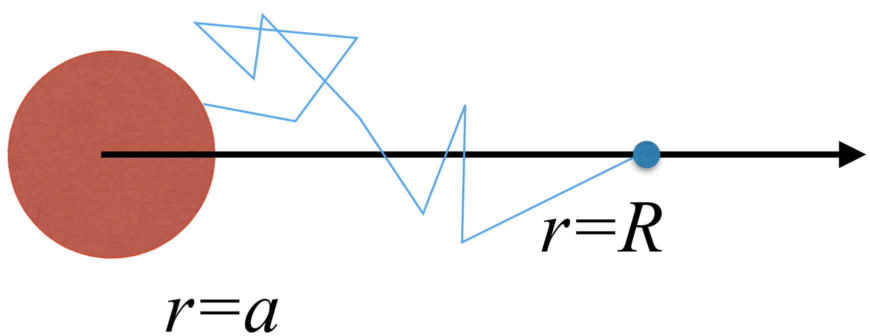


Probability of capture:



Capture in 2D

Consider a δ-function release at r=R and a circular trap of radius a.



$$u_t = D\left[u_{rr} + \frac{1}{r}u_r\right] \qquad r > 0,\ t > 0$$

$$u(r,0) = \frac{1}{2\pi R}\delta(r-R) \quad r > a$$

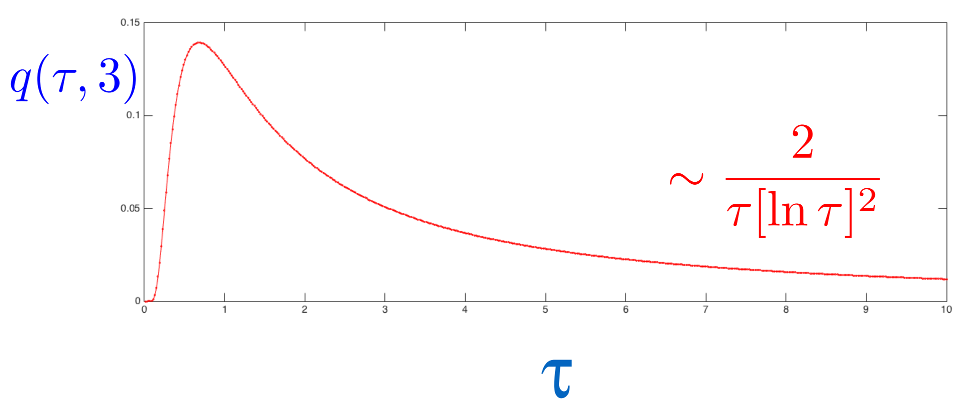
$$u(a,t) = 0 \quad t > 0$$

FPT: $p(t) = \dfrac{a^2}{d} q(\tau,\delta) \quad \delta = \dfrac{R}{a} \quad \tau = \dfrac{a^2 t}{d}$

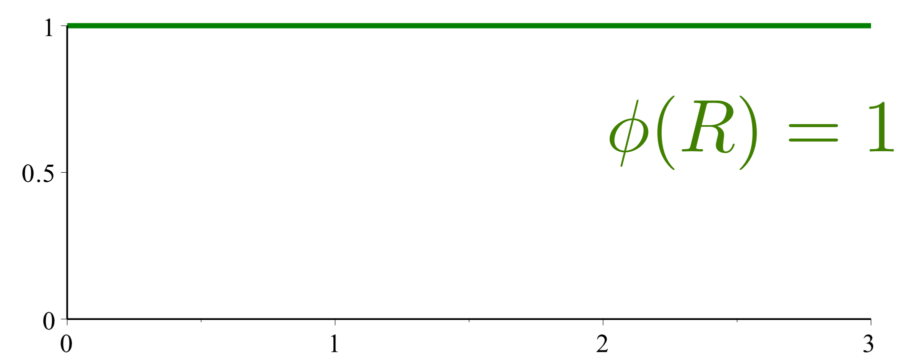
Obtained with a Laplace Transform

$$q(\tau,\delta) = \frac{2}{\pi}\int_{s=0}^{\infty} s e^{-s^2\tau} \frac{J_0(s)Y_0(\delta s) - J_0(\delta s)Y_0(s)}{[J_0(s)]^2 + [Y_0(s)]^2}\, ds$$

Capture Time

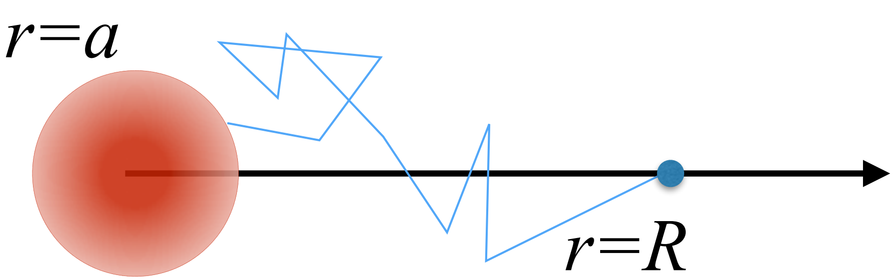


Probability of capture:



Capture in 3D

Consider a δ-function release at $r=R$ and a spherical trap of radius a.



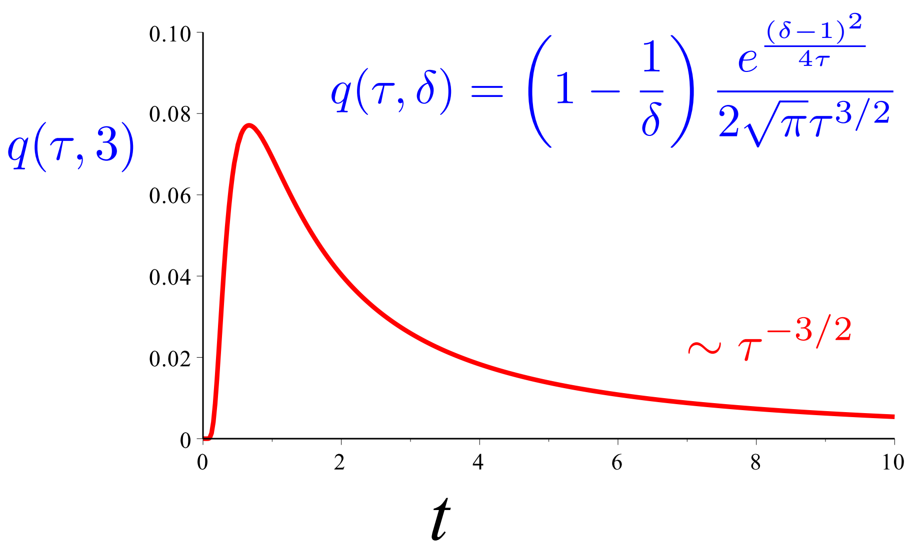
$$u_t = D\left[u_{rr} + \frac{2}{r}u_r\right] \qquad r>0,\ t>0$$

$$u(r,0) = \frac{1}{4\pi R^2}\delta(r-R) \quad r>a$$

$$u(a,t)=0 \quad t>0$$

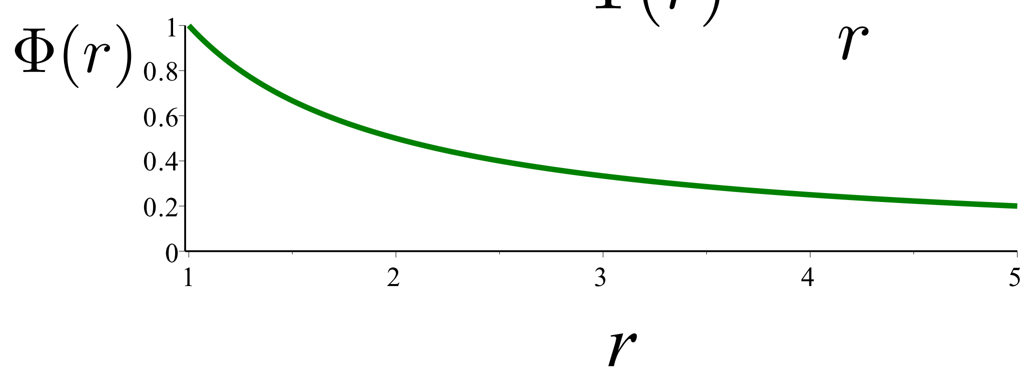
FPT: $p(t) = \frac{a^2}{d}q(\tau,\delta) \quad \delta = \frac{R}{a} \quad \tau = \frac{a^2 t}{d}$

Capture Time



Probability of capture:

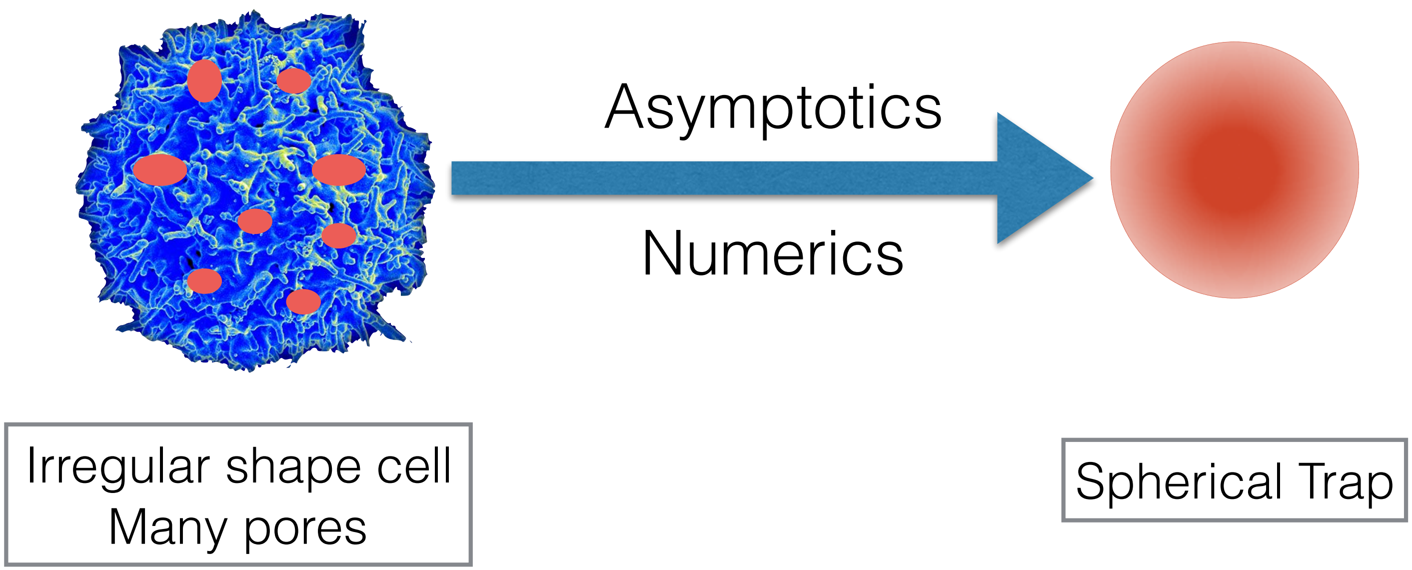
$$\Phi(r) = \frac{a}{r}$$



Homogenization & Effective Boundary Conditions

The Basic Question

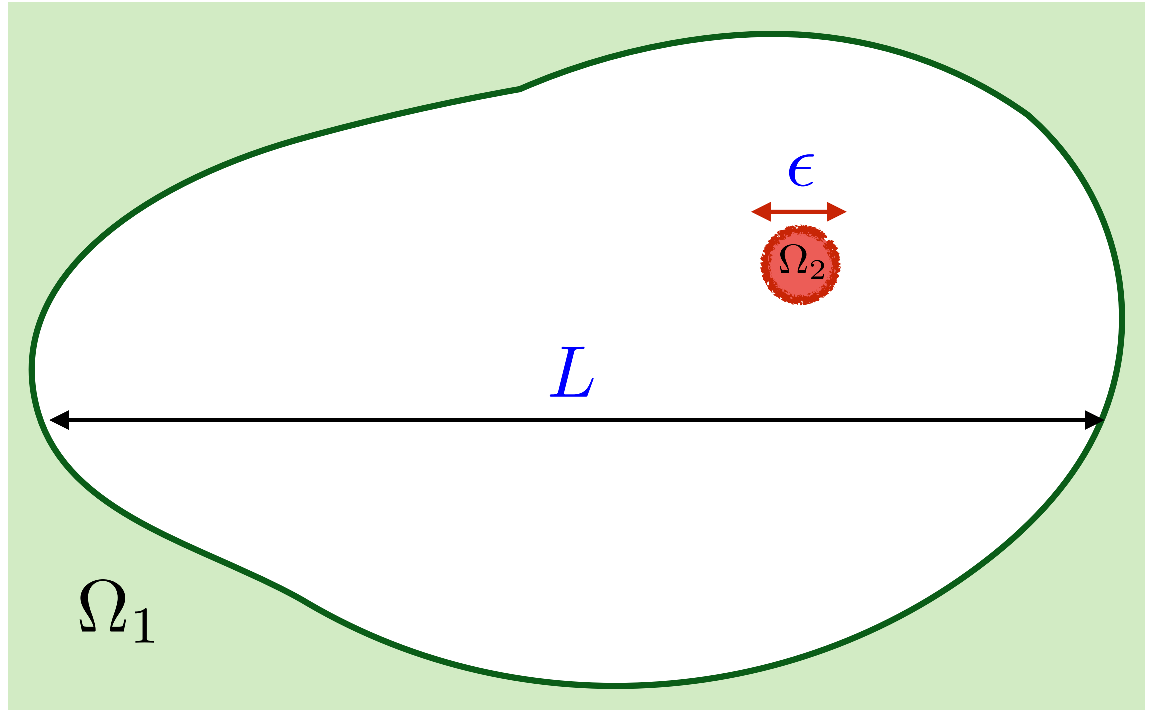
Can you replace a complicated boundary condition or geometry with a much simpler one?



The steady-state ansatz

Suppose we are looking at a small target in a large domain.

When can we assume that the density is steady near the target?



$$\rho_t = D\nabla^2\rho \text{ in } \Omega$$
$$\rho = 0 \quad \text{on } \partial\Omega_2$$
$$\nabla\rho\cdot\hat{n} = 0 \text{ on } \partial\Omega_1$$

$$L \gg \epsilon$$
$$\sqrt{Dt} \gg \epsilon$$

$$\nabla^2\rho = 0 \text{ in } \Omega$$
$$\rho = 0 \quad \text{on } \partial\Omega_2$$
$$\rho \to \rho_0 \text{ for } \epsilon \ll |\vec{x}| \ll L$$

A simple example

Sphere with a partial absorbing boundary

Smaller sphere with a totally absorbing boundary

$$\nabla^2 u = 0 \qquad r > a$$
$$u_r(a) - \kappa u(a) = 0$$
$$\lim_{r\to\infty} u(r) = C_\infty$$

$$\nabla^2 u = 0 \qquad r > b$$
$$u(b) = 0$$
$$\lim_{r\to\infty} u(r) = C_\infty$$

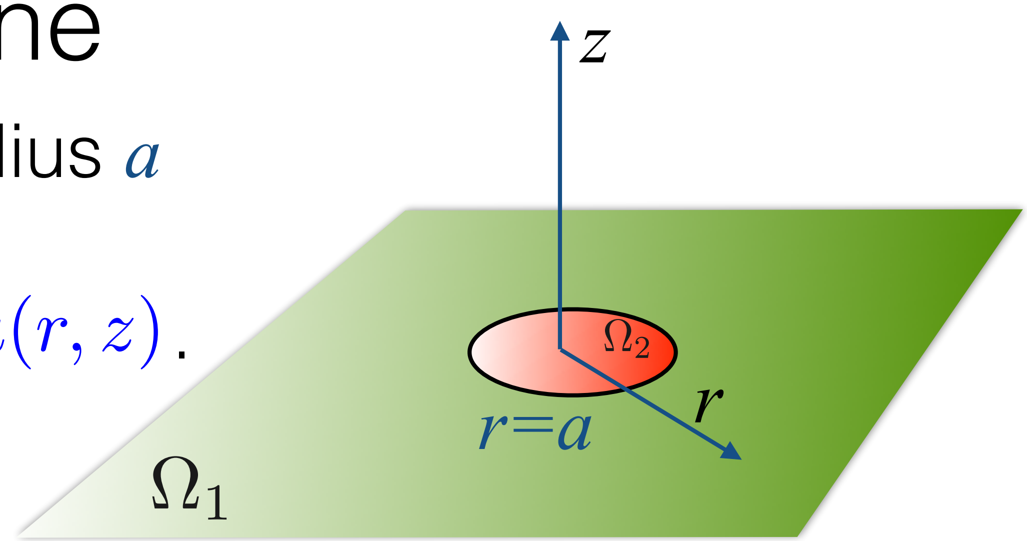
$$\boxed{u(r) = C_\infty \left(1 - \frac{1}{1 + 1/\kappa a}\,\frac{a}{r}\right)}$$

$$\boxed{u(r) = C_\infty \left(1 - \frac{b}{r}\right)}$$

Problems are equivalent when $b = \dfrac{a}{1 + 1/\kappa a}$

A Pore on a Plane

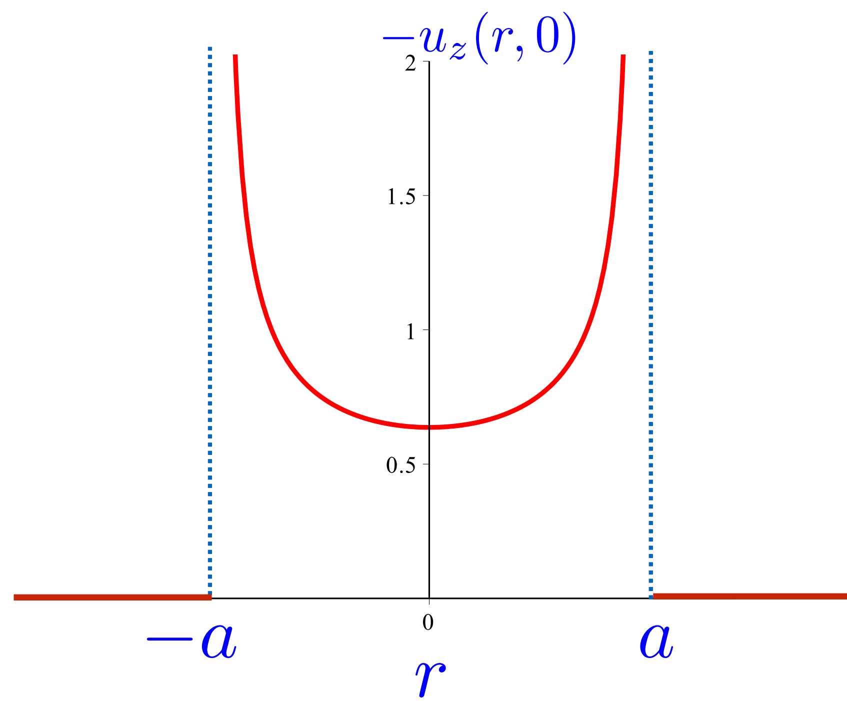
Consider a circular pore of radius a on an infinite plane.
There is an exact solution for $u(r,z)$.



$$\nabla^2 u = 0 \qquad z > 0$$
$$u_z(r,0) = 0 \text{ for } r > a$$
$$u(r,0) = 0 \text{ for } r < a$$
$$\lim_{z\to\infty} u(r,z) = C_\infty$$

On the surface, the flux is:

$$-u_z(r,0) = \begin{cases} 0 & r > a \\ \frac{2}{\pi}\frac{C_\infty}{\sqrt{a^2-r^2}} & r < a \end{cases}$$



Sneddon, Elements of PDE, (1957).

The total flux into the pore is:

$$\mathcal{F}_{pore} = 4aC_\infty$$

Flux ~ Perimeter !!

Homogenization: Many pores on a planes

For N pores of radius a:

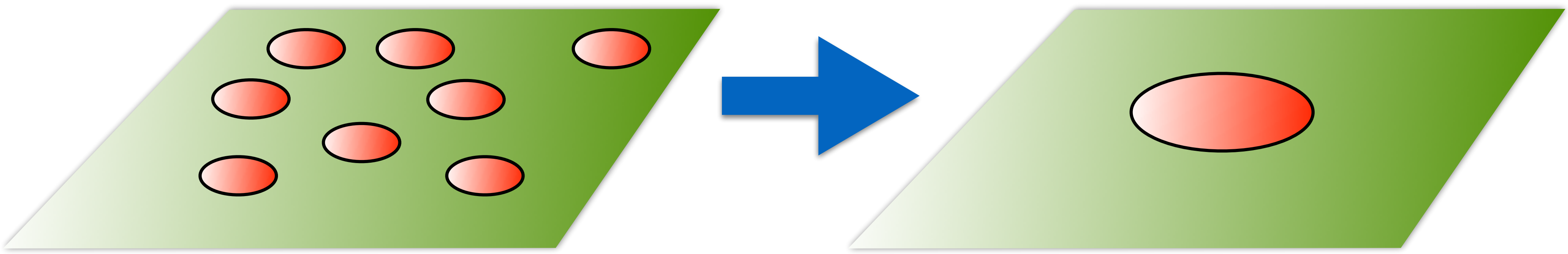
$$\mathcal{F}_{pore} = 4aNC_\infty + \frac{16a^2}{\pi}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{d_{ij}}$$

Berg & Purcell 1977

Bernoff & Lindsay 2016

When the distance between pores, $d_{ij} \gg a$.

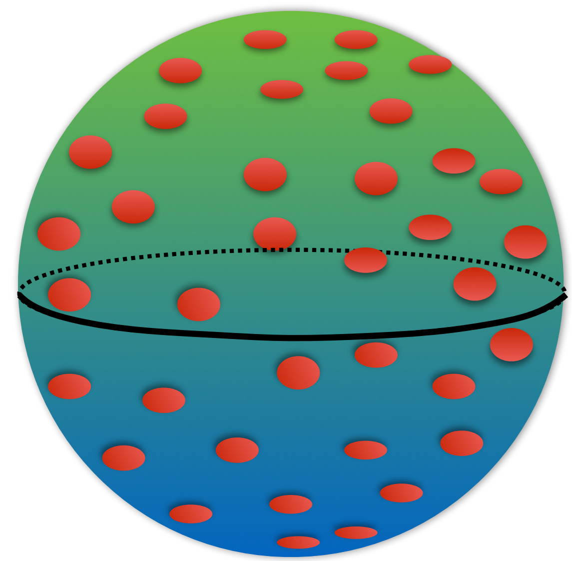
This tells us how to replace many pores with one.



Homogenization: Pores on a Sphere

This is a classic problem considered by Berg & Purcell (1977) in the dilute pore limit.

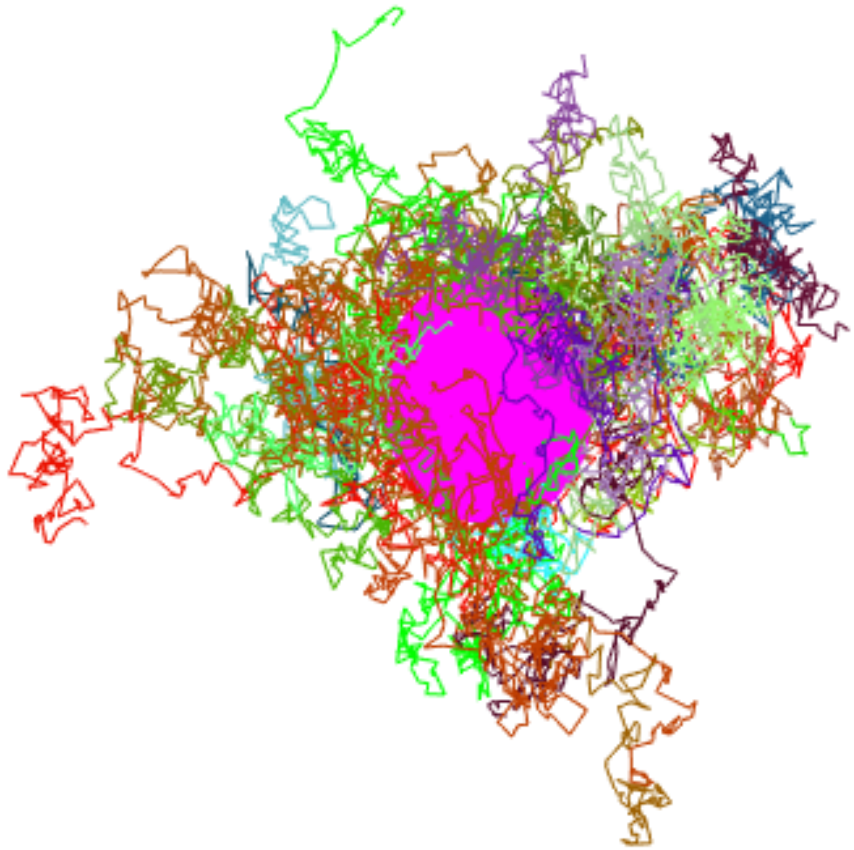
$$\mathcal{F}_{total} = 4aDNC_{\infty} \cdot \frac{1}{1 + Na/\pi R}$$



Knowing the flux allows you to replace the sphere with either a smaller sphere or a Robin BC.

The next order correction in the asymptotics has been computed by Lindsay & Ward (2016).

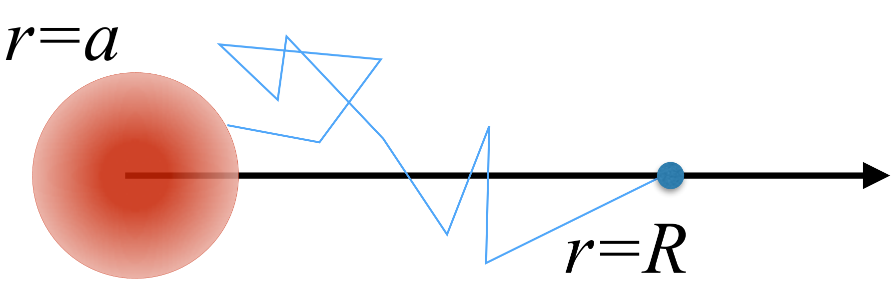
Random Walk Numerics



random walk simulation for an absorbing sphere

3D Random Walk Numerics

Consider a δ-function release at $r=R$ and a spherical trap of radius a.



$$u_t = D\left[u_{rr} + \frac{2}{r}u_r\right] \qquad r>0,\ t>0$$

$$u(r,0) = \frac{1}{4\pi R^2}\delta(r-R) \quad r>a$$

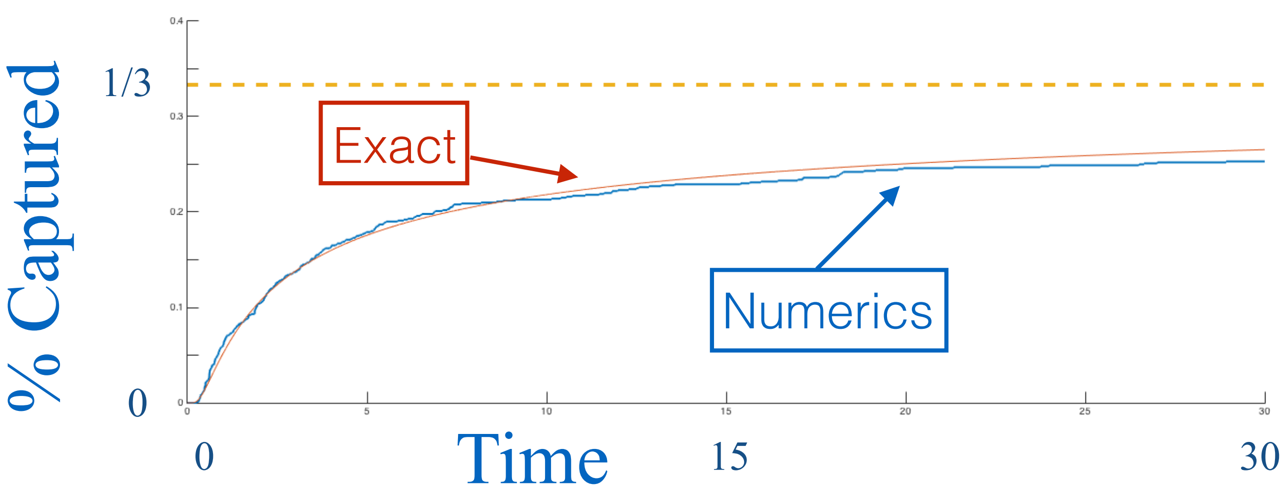
$$u(a,t) = 0 \quad t>0$$

Numerical Method:

- N particles
- Take Gaussian distributed steps of $\text{Variance} = D\Delta t$
- Stop when particle is inside target.

$$a = 1, R = 3, D = 1$$

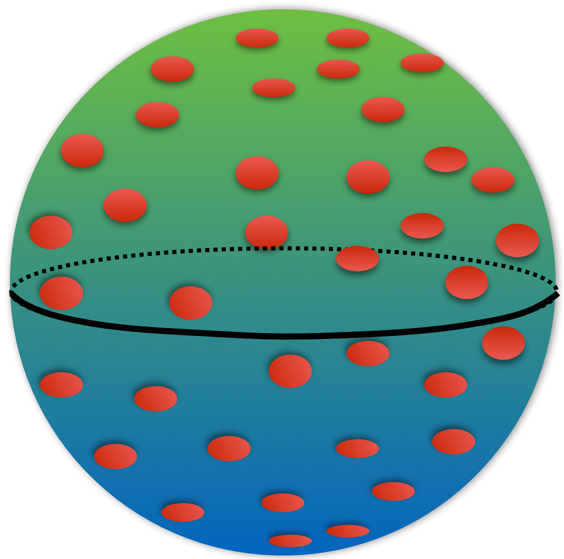
$$N = 1000, \Delta t = .001$$



Challenges:

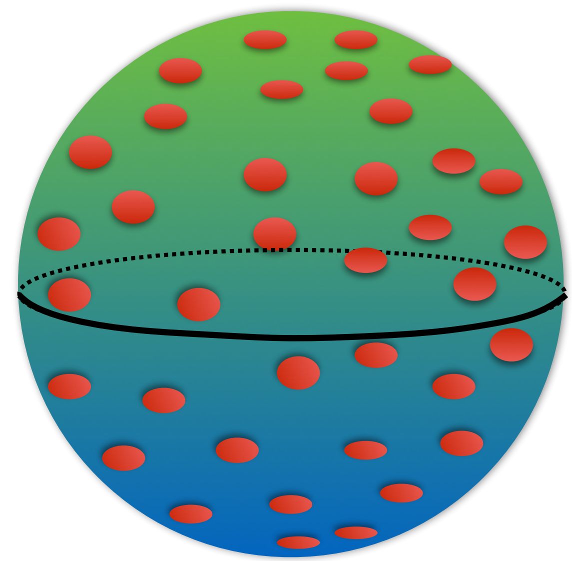
- Slow convergence.
- Long-time asymptotics often impossible to verify.
- Good for capture %, but difficult to obtain capture rate.

Challenge Problems



A Challenge Problem

Suppose a set of particles are released at the center of a sphere of radius R. The sphere has N pores of radius a on its surface. Compute the distribution of exit times for the sphere.



Notes:

- You can do this fairly effectively with a random walk particle code.

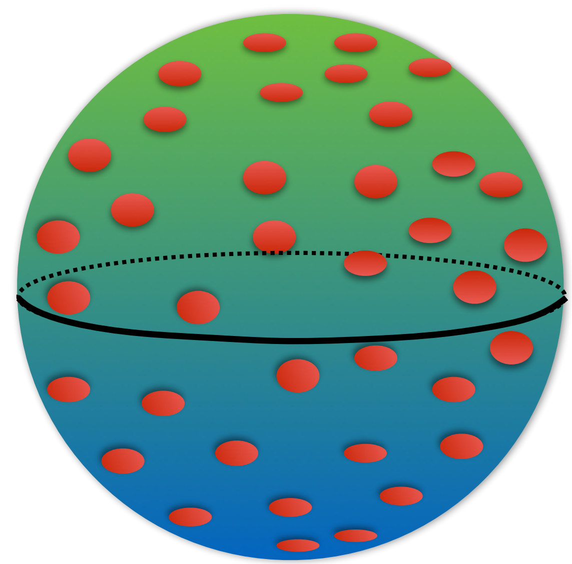
The Homogenized Challenge Problem

Suppose a set of particles are released at the center of a sphere of radius *R*. The exit of the particles is modeled by a *Robin* boundary condition. Compute the distribution of exit times for the sphere.

$$u_t = D\left[u_{rr} + \frac{2}{r}u_r\right] \qquad R > r > 0,\ t > 0$$

$$u(r,0) = \frac{1}{4\pi r^2}\delta(r) \qquad R > r > 0,$$

$$u_r(R,t) + \kappa u(R,t) = 0 \qquad t > 0$$



Notes:

- You may wish to start out with the 1D version.
- Make sure you consider κ, D small and large.

References

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