

## SOLUTIONS

1. Evaluate each of the following limits, if it exists. Assign, if possible,  $+\infty$  or  $-\infty$  to any limit that does not exist.

(a)  $\lim_{x \rightarrow -3} \frac{1 - \sqrt{10 + 3x}}{3 + x} = \frac{0}{0}$  **IDF**. Rationalize:

$$\begin{aligned} \lim_{x \rightarrow -3} \frac{1 - \sqrt{10 + 3x}}{3 + x} &= \lim_{x \rightarrow -3} \frac{1 - \sqrt{10 + 3x}}{3 + x} \left( \frac{1 + \sqrt{10 + 3x}}{1 + \sqrt{10 + 3x}} \right) \\ &= \lim_{x \rightarrow -3} \frac{1 - 10 - 3x}{(3 + x)(1 + \sqrt{10 + 3x})} \\ &= \lim_{x \rightarrow -3} \frac{-3(3 + x)}{(3 + x)(1 + \sqrt{10 + 3x})} = -\frac{3}{2}. \end{aligned}$$

(b)  $\lim_{x \rightarrow -4} \frac{x^2 - 16}{2x^2 + 15x + 28} = \frac{0}{0}$  **IDF**. Factor and cancel:

$$\lim_{x \rightarrow -4} \frac{x^2 - 16}{2x^2 + 15x + 28} = \lim_{x \rightarrow -4} \frac{(x + 4)(x - 4)}{(x + 4)(2x + 7)} = \lim_{x \rightarrow -4} \frac{x - 4}{2x + 7} = \frac{-8}{-1} = 8.$$

(c)  $\lim_{t \rightarrow 0} \frac{\sin^2(3t)}{t^2} = \frac{0}{0}$  **IDF**. Use special limits:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{\sin^2(3t)}{t^2} &= \lim_{t \rightarrow 0} \left( \frac{\sin(3t)}{t} \right)^2 = \lim_{t \rightarrow 0} \left( \frac{\sin(3t)}{t} \cdot \frac{3}{3} \right)^2 = \lim_{t \rightarrow 0} (3^2) \left( \frac{\sin(3t)}{3t} \right)^2 \\ &= 9 \left( \lim_{t \rightarrow 0} \frac{\sin(3t)}{3t} \right)^2 = 9(1)^2 = 9. \end{aligned}$$

2. Consider the following function:

$$g(x) = \begin{cases} \frac{|x - 2|}{x - 2}, & x < 2 \\ -1, & x = 2 \\ \frac{-8(2 - x)}{x^3 - 8x^2 + 12x}, & x > 2 \end{cases}$$

First note that, for  $x < 2$ ,  $|x - 2| = -(x - 2)$ , and so for  $x < 2$  the function  $g$  becomes

$$g(x) = \frac{|x - 2|}{x - 2} = \frac{-(x - 2)}{x - 2} = -1.$$

- (a) To find the vertical asymptotes we set the denominator equal to zero. Note that we only have one denominator, which occurs in the definition for  $g$  when  $x > 2$ :

$$x^3 - 8x^2 + 12x = 0 \quad \Rightarrow \quad x(x^2 - 8x + 12) = 0 \quad \Rightarrow \quad x(x - 2)(x - 6) = 0,$$

and so the candidates for vertical asymptotes are  $x = 0$ ,  $x = 2$  and  $x = 6$ . Note that  $x = 0$  does not fall in the domain  $x > 2$ , and so we neglect this value. Also, since  $g$  is defined at  $x = 2$ , i.e.  $g(2) = -1$  given, it follows that  $x = 2$  cannot be a vertical asymptote. So we use limits to check for determinate form:

$$\lim_{x \rightarrow 6} g(x) = \lim_{x \rightarrow 6} \frac{-8(2 - x)}{x^3 - 8x^2 + 12x} = \frac{32}{0} \quad \mathbf{DF}.$$

Next, use one-sided limits to verify  $x = 6$  is a vertical asymptote:

$$\lim_{x \rightarrow 6^-} g(x) = \lim_{x \rightarrow 6^-} \frac{-8(2 - x)}{x(x - 2)(x - 6)} = -\infty \quad (\text{use } x = 5.9)$$

Hence the only vertical asymptote for  $g(x)$  is  $x = 6$ .

(b) Use the definition of continuity to check for continuity of  $g$  at  $x = 2$ :

1.  $g(2) = -1$ , which is defined.
2.  $\lim_{x \rightarrow 2} g(x) = ?$  Use one-sided limits:

$$\lim_{x \rightarrow 2^-} g(x) = \lim_{x \rightarrow 2^-} (-1) = -1, \quad \lim_{x \rightarrow 2^+} g(x) = \lim_{x \rightarrow 2^+} \frac{-8(2-x)}{x^3 - 8x^2 + 12x} = \frac{0}{0} \quad \text{IDF.}$$

To see if the right limit exists, we factor and cancel a common factor:

$$\lim_{x \rightarrow 2^+} \frac{-8(2-x)}{x^3 - 8x^2 + 12x} = \lim_{x \rightarrow 2^+} \frac{-8(2-x)}{x(x-2)(x-6)} \lim_{x \rightarrow 2^+} \frac{8}{x(x-6)} = \frac{8}{-8} = -1.$$

Since  $\lim_{x \rightarrow 2^-} g(x) = -1 = \lim_{x \rightarrow 2^+} g(x)$ , it follows that the limit exists and  $\lim_{x \rightarrow 2} g(x) = -1$ .

3. Note that  $g(2) = -1 = \lim_{x \rightarrow 2} g(x)$ . Since all three properties are satisfied, it follows from the definition of continuity at a point that  $g(x)$  is continuous at  $x = 2$ .

3. Let  $f(x) = \sqrt{5 - 2x}$ .

(a) Use the definition of the derivative to find  $f'(x)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{5 - 2(x+h)} - \sqrt{5 - 2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{5 - 2x - 2h} - \sqrt{5 - 2x}}{h} \left( \frac{\sqrt{5 - 2x - 2h} + \sqrt{5 - 2x}}{\sqrt{5 - 2x - 2h} + \sqrt{5 - 2x}} \right) \\ &= \lim_{h \rightarrow 0} \frac{5 - 2x - 2h - (5 - 2x)}{h(\sqrt{5 - 2x - 2h} + \sqrt{5 - 2x})} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{h(\sqrt{5 - 2x - 2h} + \sqrt{5 - 2x})} = \frac{-2}{2\sqrt{5 - 2x}} = -\frac{1}{\sqrt{5 - 2x}} \end{aligned}$$

(b) Find the equation of the tangent line to  $f(x)$  at  $x = \frac{1}{2}$

**Solution:** To find the equation of the tangent line to  $f(x)$  at  $x = \frac{1}{2}$ , we need the slope  $m$  at this point and a point on the line. With this information, we use the point-slope form of a line to find the equation. The slope is given by the derivative at this point, namely  $f'\left(\frac{1}{2}\right)$ :

$$f'(x) = -\frac{1}{\sqrt{5 - 2x}} \quad \Rightarrow \quad f'\left(\frac{1}{2}\right) = -\frac{1}{\sqrt{5 - 1}} = -\frac{1}{2}.$$

So the slope of the tangent line to  $f(x)$  at  $x = \frac{1}{2}$  is  $m = -\frac{1}{2}$ . The point on the line is simply  $(x_0, y_0)$ , where  $x_0 = \frac{1}{2}$  and we calculate  $y_0$  by:

$$y_0 = f(x_0) = f\left(\frac{1}{2}\right) = \sqrt{5 - 2\left(\frac{1}{2}\right)} = 2.$$

Using this information we may find the equation of the tangent line:

$$\begin{aligned} y - y_0 &= m(x - x_0) \\ y - 2 &= -\frac{1}{2}\left(x - \frac{1}{2}\right) \\ y &= -\frac{1}{2}x + \frac{1}{4} + 2 \\ y &= -\frac{1}{2}x + \frac{9}{4}. \end{aligned}$$

4. Find the derivative of the following functions using the appropriate differentiation rules, making any obvious simplifications.

$$(a) \quad f(x) = \frac{x^3 + 2x - 1}{\sqrt{x}} = \frac{x^3}{x^{\frac{1}{2}}} + \frac{2x}{x^{\frac{1}{2}}} - \frac{1}{x^{\frac{1}{2}}} = x^{\frac{5}{2}} + 2x^{\frac{1}{2}} - x^{-\frac{1}{2}}$$

$$f'(x) = \frac{5}{2}x^{\frac{3}{2}} + 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} = \frac{5}{2}x^{\frac{3}{2}} + \frac{1}{x^{\frac{1}{2}}} + \frac{1}{2x^{\frac{3}{2}}} = \frac{5}{2}\sqrt{x^3} + \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{x^3}}.$$

$$(b) \quad g(t) = \frac{\cos(t)}{1 + \sin(t)}. \text{ Use the Quotient Rule:}$$

$$\begin{aligned} g'(t) &= \frac{(-\sin(t))(1 + \sin(t)) - (0 + \cos(t))\cos(t)}{(1 + \sin(t))^2} \\ &= \frac{-\sin(t) - (\sin^2(t) + \cos^2(t))}{(1 + \sin(t))^2} = \frac{-\sin(t) - 1}{(1 + \sin(t))^2} = \frac{-(1 + \sin(t))}{(1 + \sin(t))^2} = -\frac{1}{1 + \sin(t)}. \end{aligned}$$

$$(c) \quad h(x) = (2 + \tan(x))(x^2 + 7x)^2. \text{ Use the Product Rule and the Chain Rule together:}$$

$$\begin{aligned} h'(x) &= (0 + \sec^2(x))(x^2 + 7x)^2 + 2(x^2 + 7x)(2x + 7)(2 + \tan(x)) \\ &= (x^2 + 7x) [(x^2 + 7x)\sec^2(x) + 2(2x + 7)(2 + \tan(x))]. \end{aligned}$$

**Bonus Question:** The mass of glucose in a metabolic experiment decreases according to the equation

$$m(t) = 50 - 2t^{\frac{3}{2}},$$

where  $m(t)$  = mass of glucose in grams (g) at time  $t$ , and  $t$  is time in hours.

- (a) The initial mass is the mass when time  $t = 0$ , so  $m(0) = 50 - 2(0)^{\frac{3}{2}} = 50$  grams.  
 (b) To find the rate of change in the amount of glucose after 4 hours we differentiate  $m(t)$  and evaluate the derivative at  $t = 4$  hours:

$$\frac{dm}{dt} = 0 - 2\left(\frac{3}{2}\right)t^{\frac{1}{2}} = -3\sqrt{t} \implies \left.\frac{dm}{dt}\right|_{t=4} = -3\sqrt{4} = -6 \text{ grams/hour.}$$

So after 4 hours, the mass of glucose is **decreasing** at a rate of 6 grams/hour. To find the amount of glucose metabolized after 4 hours we simply subtract the amount left after 4 hours from the initial mass:

$$m(0) - m(4) = 50 - (50 - 2(4)^{\frac{3}{2}}) = 50 - (50 - 16) = 50 - 34 = 16 \text{ grams.}$$

Hence 16 grams of glucose have been metabolized after 4 hours.