MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

QUIZ #4 **Mathematics 1000** FALL 2002

SOLUTIONS

1. $f(x) = \dfrac{2x}{x^2+1}$, $\quad f'(x) = \dfrac{2(1-x^2)}{(x^2+1)^2}$, $\quad f''(x) = \dfrac{4x(x^2-3)}{(x^2+1)^3}$

   (a) Critical points are those which make $f'(x) = 0$ or undefined. $f'(x)$ is undefined when the denominator is equal to zero, i.e. $x^2+1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i$. Since the numbers are complex (imaginary), it follows that the denominator is never zero for any real numbers, so $f'(x)$ is never undefined.

   $$f'(x) = 0 \quad \Rightarrow \quad \frac{2(1-x^2)}{(x^2+1)^2} = 0 \quad \Rightarrow \quad 2(1-x^2) = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1.$$

   So the critical numbers are $x = -1$ and $x = 1$. Using these points we find the intervals on which f is increasing and decreasing, shown below. Hence f is increasing on the interval $(-1, 1)$ and decreasing on $(-\infty, -1)$ and $(1, \infty)$. Thus by the first derivative test $(-1, -1)$ is a relative minimum point, while $(1, 1)$ is a relative maximum (the y–coordinates of the extrema are found by evaluating $f(-1) = \frac{-2}{2} = -1$, and $f(1) = \frac{2}{2} = 1$). Points of inflection occur when the second derivative is undefined or zero. Note that $f''(x)$ is never undefined since the denominator is never zero. So the possible points of inflection are given by

   $$f''(x) = 0 \quad \Rightarrow \quad \frac{4x(x^2-3)}{(x^2+1)^3} \quad \Rightarrow \quad 4x(x^2-3) = 0 \quad \Rightarrow \quad 4x = 0 \quad \text{or} \quad x^2 = 3$$

   and so the possible inflection points are $x = 0$ and $x = \pm\sqrt{3}$. Using these values on a number line we see that f is concave up on the intervals $(-\sqrt{3}, 0)$ and $(\sqrt{3}, \infty)$ and concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$. Hence the points $\left(-\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$, $(0, 0)$, and $\left(\sqrt{3}, \frac{\sqrt{3}}{2}\right)$ are all inflection points since concavity changes at these points (the y–coordinates of these points are obtained by evaluating $f(x)$ at each of $x = -\sqrt{3}$, $x = 0$ and $x = \sqrt{3}$). To find x–intercepts, we set $y = 0$ and for y–intercepts we set $x = 0$:

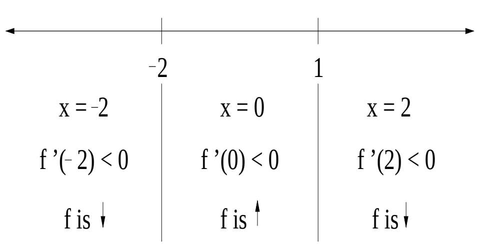
   $$0 = \frac{2x}{x^2+1} \quad \Rightarrow \quad x = 0, \qquad f(0) = \frac{2(0)}{0^2+1} = 0,$$

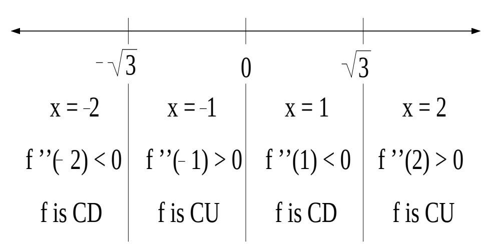
   and so the y–intercept is the origin $(0, 0)$, and it is the only x–intercept.

   (b) To find horizontal asymptotes, we check the limits as $x \to \pm\infty$:

   $$\begin{aligned} \lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x}{x^2+1} = \frac{\infty}{\infty} \\ &= \lim_{x\to\infty} \left(\frac{2x}{x^2+1}\right) \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0 \end{aligned}$$

   And so $y = 0$ is a horizontal asymptote of $f(x)$. Checking the limit as $x \to -\infty$, we get the same result. So $y = 0$ is the only horizontal asymptote.



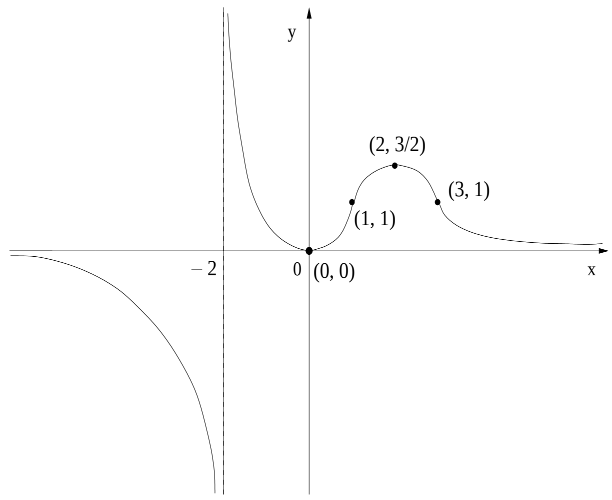


2. Sketch the graph of a function $y = f(x)$ with the following properties:

Solution: Interpret the information, and then draw the graph. Note that $x = -2$ is not in the domain of the function, and so f is undefined there (this means we either have a vertical asymptote or a hole in the graph at $x = -2$). Since we are told that

$$\lim_{x\to -2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x\to -2^+} f(x) = +\infty$$

this verifies that $x = -2$ is a vertical asymptote, and tells us how the function behaves when x is close to -2.



From the other information we can gather that f is increasing on $(0, 2)$, and decreasing on $(-\infty, -2)$, $(0, 2)$, and $(2, \infty)$, and so the point $(0, 0)$ is a relative minimum, and $(0, 2)$ is a relative maximum. Furthermore, f is concave up on the intervals $(-2, 1)$, and $(3, \infty)$, and concave down on $(-\infty, -2)$ and $(1, 3)$. Hence we have points of inflection at $x = 1$ and $x = 3$ since the concavity changes at these points. In addition, we are given the coordinates of the extrema and inflection points: $(0, 0)$, $\left(2, \frac{3}{2}\right)$, $(1, 1)$, and $(3, 1)$. Finally we are told that

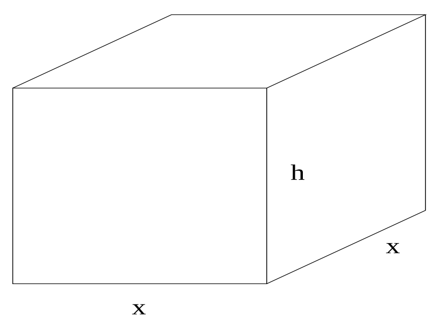
$\lim_{x\to -\infty} f(x) = \lim_{x\to\infty} f(x) = 0$ and so $y = 0$ is a horizontal asymptote of f (in both directions).

Using this information, the graph of the function f is shown in the diagram.

3. If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest volume of such a box.

Solution: Since the base of the box is a square, let x be the width and length of the box and let h be the height, as shown in the diagram. We want to maximize the volume of the box, which is given by the primary equation

$$V = x \cdot x \cdot h = x^2 h$$



We need to express this primary equation in terms of a single variable, say x. To do this we use the fact that there is 1200 cm^2 of material available to make the box to generate a secondary equation — that is, the surface area of the box is 1200 cm^2. Since the box has an open top the surface area is given by

$$S_A = x^2 + 4xh \quad \Leftrightarrow \quad 1200 = x^2 + 4xh \quad \Rightarrow \quad h = \frac{1200 - x^2}{4x} = \frac{300}{x} - \frac{x}{4}$$

Substituting this into the primary equation, we get a function of a single variable $V = V(x)$:

$$V(x) = x^2\left(\frac{300}{x} - \frac{x}{4}\right) = 300x - \frac{x^3}{4} \qquad \text{Domain: } x > 0$$

To find the critical numbers we set the first derivative to zero:

$$V'(x) = 300 - \frac{3}{4}x^2 = 0 \quad \Rightarrow \quad x^2 = 400 \quad \Rightarrow \quad x = \pm 20\,\text{cm}$$

where we reject the negative since we require $x > 0$. We verify this value of x gives a maximum for $V(x)$ by using the second derivative test:

$V''(x) = -\frac{3}{2}x \quad \Rightarrow \quad V''(20) = -30 < 0$ which implies that $x = 20$ is a relative maximum.

Hence the largest volume of a box with a square base that can be made out of 1200 cm^2 of material is

$$V(20) = 300(20) - \frac{20^3}{4} = 4000\,\text{cm}^3.$$