MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

PRACTICE PROBLEMS | **Mathematics 1000** | FALL 2002

Solutions to Worksheet

1. Find the derivatives for :

(a) $f(x) = \sqrt{\cos(x^2 e^{-x})} = \left(\cos\left(x^2 e^{-x}\right)\right)^{\frac{1}{2}}$. Differentiate using the chain and product rules:

$$\begin{aligned} f'(x) &= \frac{1}{2}\left(\cos\left(x^2e^{-x}\right)\right)^{-\frac{1}{2}} \cdot \left(-\sin\left(x^2e^{-x}\right)\right) \cdot \left(2xe^{-x} + \left(e^{-x}\cdot -1\right)x^2\right) \\ &= -\frac{1}{2}e^{-x}\left(2x - x^2\right)\left(\cos\left(x^2e^{-x}\right)\right)^{-\frac{1}{2}}\sin\left(x^2e^{-x}\right) \\ &= \frac{1}{2}x(x-2)e^{-x}\frac{\sin\left(x^2e^{-x}\right)}{\sqrt{\cos\left(x^2e^{-x}\right)}} \end{aligned}$$

(b) $f(x) = \sin^2\left(x^{\frac{1}{3}} + \ln(x)\right) = \left[\sin\left(x^{\frac{1}{3}} + \ln(x)\right)\right]^2$. Differentiate using the chain rule:

$$\begin{aligned} f'(x) &= 2\left[\sin\left(x^{\frac{1}{3}} + \ln(x)\right)\right] \cdot \cos\left(x^{\frac{1}{3}} + \ln(x)\right) \cdot \left(\frac{1}{3x^{\frac{2}{3}}} + \frac{1}{x}\right) \\ &= 2\sin\left(x^{\frac{1}{3}} + \ln(x)\right)\cos\left(x^{\frac{1}{3}} + \ln(x)\right) \cdot \frac{x^{\frac{1}{3}} + 3}{3x} \\ &= \frac{2\left(x^{\frac{1}{3}} + 3\right)}{3x}\sin\left(x^{\frac{1}{3}} + \ln(x)\right)\cos\left(x^{\frac{1}{3}} + \ln(x)\right) \end{aligned}$$

(c) $f(x) = \dfrac{2x+1}{\sqrt{x^2+1}}$. Differentiate using the quotient and chain rules:

$$\begin{aligned} f'(x) &= \frac{2\sqrt{x^2+1} - (2x+1)\frac{1}{2}\left(x^2+1\right)^{-\frac{1}{2}} \cdot 2x}{\left[\sqrt{x^2+1}\right]^2} \\ &= \frac{2\sqrt{x^2+1} - \frac{x(2x+1)}{\sqrt{x^2+1}}}{x^2+1} \\ &= \frac{\frac{2x^2+2-2x^2-x}{\sqrt{x^2+1}}}{x^2+1} \\ &= \frac{2-x}{\left(x^2+1\right)^{\frac{3}{2}}} \end{aligned}$$

(d) $f(x) = \ln\left|\cot(x) + \csc(x)\right|$. Differentiate using the chain rule:

$$f'(x) = \frac{1}{\cot(x)+\csc(x)} \cdot \left(-\csc^2(x) - \csc(x)\cot(x)\right) = \frac{-\csc(x)\left(\csc(x)+\cot(x)\right)}{\cot(x)+\csc(x)} = -\csc(x)$$

(e) $f(t) = 2^t\sec(2t)$. Differentiate using the product and chain rules:

$$f'(t) = 2^t \cdot \ln(2)\sec(2t) + 2^t\sec(2t)\tan(2t)\cdot 2 = 2^t\sec(2t)\left(\ln(2) + 2\tan(2t)\right)$$

(f) $f(x) = \log_3(x^3 + 2x^2 - 1)$. Differentiate using the chain rule:

$$f'(x) = \frac{1}{\ln(3)\left(x^3+2x^2-1\right)} \cdot \left(3x^2+4x\right) = \frac{3x^2+4x}{\ln(3)\left(x^3+2x^2-1\right)}$$

(g) $y = \dfrac{(x^6+1)^3(x^4+1)}{14\sqrt{x+9}}$. Use logarithmic differentiation:

$$\begin{aligned}
\ln(y) &= \ln\left[\frac{(x^6+1)^3(x^4+1)}{14\sqrt{x+9}}\right] \\
\ln(y) &= 3\ln(x^6+1) + \ln(x^4+1) - \left(\ln(14) + \frac{1}{2}\ln(x+9)\right) \\
\frac{1}{y}\frac{dy}{dx} &= 3\frac{1}{x^6+1}\cdot(6x^5+0) + \frac{1}{x^4+1}\cdot(4x^3+0) - 0 - \frac{1}{2}\frac{1}{x+9}\cdot(1) \\
\frac{dy}{dx} &= y\left[\frac{18x^5}{x^6+1} + \frac{4x^3}{x^4+1} - \frac{1}{2(x+9)}\right] \\
\frac{dy}{dx} &= \frac{(x^6+1)^3(x^4+1)}{14\sqrt{x+9}}\left[\frac{18x^5}{x^6+1} + \frac{4x^3}{x^4+1} - \frac{1}{2(x+9)}\right]
\end{aligned}$$

(h) $f(x) = (1-x)^{\frac{1}{x}}$. Use logarithmic differentiation:

$$\begin{aligned}
\ln(f(x)) &= \ln\left[(1-x)^{\frac{1}{x}}\right] \\
\ln(f(x)) &= \frac{1}{x}\ln(1-x) = \frac{\ln(1-x)}{x} = \ln(1-x)\cdot x^{-1} \\
\frac{1}{f(x)}f'(x) &= \frac{1}{1-x}\cdot(-1)x^{-1} + (-1)x^{-2}\ln(1-x) \\
\frac{1}{f(x)}f'(x) &= -\frac{1}{x(1-x)} - \frac{\ln(1-x)}{x^2} \\
f'(x) &= f(x)\left[-\frac{1}{x}\left(\frac{1}{(1-x)} + \frac{\ln(1-x)}{x}\right)\right] \\
f'(x) &= -\frac{(1-x)^{\frac{1}{x}}}{x}\left[\frac{1}{(1-x)} + \frac{\ln(1-x)}{x}\right]
\end{aligned}$$

2. Find $\dfrac{dy}{dx}$ given that $x + 2y = \tan\left(x^3y\right)$. Differentiate both sides implicitly:

$$\begin{aligned}
1 + 2\frac{dy}{dx} &= \sec^2\left(x^3y\right)\cdot\left(3x^2y + x^3\frac{dy}{dx}\right) \\
1 + 2\frac{dy}{dx} &= 3x^2y\sec^2\left(x^3y\right) + x^3\sec^2\left(x^3y\right)\frac{dy}{dx} \\
2\frac{dy}{dx} - x^3\sec^2\left(x^3y\right)\frac{dy}{dx} &= 3x^2y\sec^2\left(x^3y\right) - 1 \\
\frac{dy}{dx}\left[2 - x^3\sec^2\left(x^3y\right)\right] &= 3x^2y\sec^2\left(x^3y\right) - 1 \\
\frac{dy}{dx} &= \frac{3x^2y\sec^2\left(x^3y\right) - 1}{2 - x^3\sec^2\left(x^3y\right)}
\end{aligned}$$

3. (a) $y = e^{x^5y}$. Differentiate implicitly:

$$\begin{aligned}
\frac{dy}{dx} &= e^{x^5y}\cdot\left(5x^4y + x^5\frac{dy}{dx}\right) \\
\frac{dy}{dx} &= 5x^4ye^{x^5y} + x^5e^{x^5y}\frac{dy}{dx} \\
\frac{dy}{dx}\left[1 - x^5e^{x^5y}\right] &= 5x^4ye^{x^5y} \\
\frac{dy}{dx} &= \frac{5x^4ye^{x^5y}}{1 - x^5e^{x^5y}}
\end{aligned}$$

(b) $(2x+y)^{\frac{3}{2}} = x^2+y^2$. Differentiate implicitly

$$\begin{aligned}
\frac{3}{2}(2x+y)^{\frac{1}{2}} \cdot \left(2+\frac{dy}{dx}\right) &= 2x+2y\frac{dy}{dx} \\
3(2x+y)^{\frac{1}{2}}\left(2+\frac{dy}{dx}\right) &= 4x+4y\frac{dy}{dx} \\
6(2x+y)^{\frac{1}{2}} + 3(2x+y)^{\frac{1}{2}}\frac{dy}{dx} &= 4x+4y\frac{dy}{dx} \\
\frac{dy}{dx}\left[3(2x+y)^{\frac{1}{2}} - 4y\right] &= 4x - 6(2x+y)^{\frac{1}{2}} \\
\frac{dy}{dx} &= \frac{4x-6(2x+y)^{\frac{1}{2}}}{3(2x+y)^{\frac{1}{2}}-4y}
\end{aligned}$$

(c) $\sin^2(x+y) = x+4y \Rightarrow [\sin(x+y)]^2 = x+4y$. Differentiate implicitly:

$$\begin{aligned}
2\sin(x+y)\cdot\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) &= 1+4\frac{dy}{dx} \\
2\sin(x+y)\cos(x+y) + 2\sin(x+y)\cos(x+y)\frac{dy}{dx} &= 1+4\frac{dy}{dx} \\
\frac{dy}{dx}\left[2\sin(x+y)\cos(x+y)-4\right] &= 1-2\sin(x+y)\cos(x+y) \\
\frac{dy}{dx} &= \frac{1-2\sin(x+y)\cos(x+y)}{2\sin(x+y)\cos(x+y)-4}
\end{aligned}$$

4. Given that $x^3+y^3=12$ prove that $\dfrac{d^2y}{dx^2} = -\dfrac{24x}{y^5}$. Differentiate implicitly twice:

$$\begin{aligned}
3x^2+3y^2\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{x^2}{y^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d^2y}{dx^2} &= -\frac{\left(2xy^2-2x^2y\frac{dy}{dx}\right)}{(y^2)^2} \\
\frac{d^2y}{dx^2} &= \frac{-2xy^2+2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4} \\
\frac{d^2y}{dx^2} &= \frac{-2xy^2-\frac{2x^4}{y}}{y^4} \\
\frac{d^2y}{dx^2} &= \frac{-2xy^3-2x^4}{y^5} \\
\frac{d^2y}{dx^2} &= \frac{-2x(y^3+x^3)}{y^5} = \frac{-2x(12)}{y^5} = -\frac{24x}{y^5}
\end{aligned}$$

5. Let P, T, and V be the pressure, temperature, and volume of the gas at a given time t. The gas is being compressed (i.e. decreasing the volume) and heated (increasing the temperature), and so the pressure of the gas should be expected to rise, i.e. we expect $\frac{dP}{dt} > 0$. We are given that

$$\frac{dT}{dt} = 2\,\mathrm{K/s} \qquad \text{and} \qquad \frac{dV}{dt} = -0.1 = -\frac{1}{10}\,\mathrm{L/s}$$

(where L is in litres, T is in kelvins, and s is time in seconds) and are asked to find $\frac{dP}{dt}$ when the temperature of the gas is 300 K and the volume is 10 L. We are told that the equation relating the variables is $PV = 5T$ and so we differentiate with respect to time, using the product rule for the PV term:

$$V\frac{dP}{dt} + P\frac{dV}{dt} = 5\frac{DT}{dt} \quad \Rightarrow \quad \frac{dP}{dt} = \frac{5\frac{dT}{dt} - P\frac{dV}{dt}}{V}$$

The only piece of information we don't have is the pressure P when $T = 300$ K and $V = 10$ L, but we know that

$$PV = 5T \quad \Rightarrow \quad P = \frac{5T}{V} = \frac{5(300)}{10} = 150\,\mathrm{kPa}$$

and so the pressure is 150 at the desired time. We fill in the information to find that the pressure of the gas is increasing at a rate of

$$\frac{dP}{dt} = \frac{5(2) - (150)(-\frac{1}{10})}{10} = \frac{25}{10} = \frac{5}{2}\,\text{kPa/s}$$

when the temperature is 300 K and the volume is 10 L.

6. Let $x =$ the distance from the plane to the point 10 km directly above the radar station, $y = 10$ km the altitude of the plane as it flies horizontally (fixed length), and $z =$ the distance from the radar station to the plane at some time t. We are given that the distance between the plane and the station is increasing at a rate of $\frac{dz}{dt} = 480$ km/h, and are asked to find the speed of the plane $\frac{dx}{dt}$ when the plane is 26 km from the radar station. Since this situation sets up a right triangle, the equation relating the variables is

$$x^2 + 10^2 = z^2 \quad \Rightarrow \quad 2x\frac{dx}{dt} + 0 = 2z\frac{dz}{dt} \quad \Rightarrow \quad \frac{dx}{dt} = \frac{z}{x}\frac{dz}{dt}$$

When $z = 26$ km, $x = \sqrt{z^2 - 10^2} = \sqrt{26^2 - 10^2} = 24$, and so the speed of the plane is

$$\frac{dx}{dt} = \frac{26\,\text{km}}{24\,\text{km}}(480\,\text{km/h}) = 520\,\text{km/h}$$

when the plane is 26 km from the radar station.

7. Let $V =$ the volume of the balloon in cm^3, $S =$ the surface area of the balloon in cm^2, and $r =$ the radius of the balloon in cm at some time t. We are told that the surface area is increasing at a rate of 20 cm^2/s, i.e. $\frac{dS}{dt} = 20$ cm^2/s, when the radius $r = 10$ cm (the time is in seconds (s)), and are asked to find $\frac{dV}{dt}$ at this instant. The variables are related by the equations

$$V = \frac{4}{3}\pi r^3 \qquad \text{and} \qquad S = 4\pi r^2$$

Differentiating gives

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \qquad \text{and} \qquad \frac{dS}{dt} = 8\pi r\frac{dr}{dt}.$$

Since $\frac{dS}{dt} = 20$, we can solve for $\frac{dr}{dt}$ and substitute it into the first derivative to solve the question:

$$\frac{dr}{dt} = \frac{1}{8\pi r}\frac{dS}{dt} = \frac{1}{8\pi r}(20) = \frac{5}{2\pi r} \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2\frac{5}{2\pi r} = 2(5)r = 2(5)(10) = 100,$$

and so the volume is increasing at a constant rate of 100 cm^3/s, as desired.

8. Let x be the horizontal distance from the base of the ladder to the wall, y be the vertical distance from the top of the ladder to the ground, $\ell = 25$ ft be the length of the ladder (a fixed length), and θ be the angle between the ladder and the ground at some time t.

(a) We are given that $\frac{dx}{dt} = 3$ ft/sec, and are asked to find $\frac{dy}{dt}$ when $y = 20$ ft. The variables are related using Pythagoras

$$x^2 + y^2 = 25^2 \quad \Rightarrow \quad 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \Rightarrow \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $y = 20$ ft, $x = \sqrt{25^2 - 20^2} = 15$ ft, and so $\frac{dy}{dt} = -\frac{15}{20}(3\,\text{ft/sec}) = -\frac{9}{4}$ ft/sec. That is, the top of the ladder is sliding down at a rate of $\frac{9}{4}$ ft/sec when the top of the ladder is 20 ft above the ground.

(b) For part (b), we want to find $\frac{d\theta}{dt}$ when $y = 20$ ft, and so we can relate the angle between the ladder and ground and the height of the top of the ladder using trigonometry:

$$\sin(\theta) = \frac{y}{25} \quad \Rightarrow \quad \cos(\theta)\frac{d\theta}{dt} = \frac{1}{25}\frac{dy}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{25\cos(\theta)}\frac{dy}{dt}$$

From part (a), we know that when $y = 20$, $x = 15$ and so $\cos(\theta) = \frac{15}{25} = \frac{3}{5}$, and we found in (a) that $\frac{dy}{dt} = -\frac{9}{4}$ ft/sec. Hence

$$\frac{d\theta}{dt} = \frac{1}{25\left(\frac{3}{5}\right)}\left(-\frac{9}{4}\right) = \frac{1}{15}\cdot -\frac{9}{4} = -\frac{3}{20}\,\text{rads/sec}$$

That is, the angle between the ladder and ground is decreasing a rate of $\frac{3}{20}$ rads/sec when the top of the ladder is 20 feet from the ground.