MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 6 **Mathematics 1000** FALL 2002

SOLUTIONS

1. $f(x) = x^5 - 2x^4$.

   (a) Find the first and second derivatives:

$$\begin{aligned} f'(x) &= 5x^4 - 8x^3 = x^3(5x - 8) \\ f''(x) &= 20x^3 - 24x^2 = 4x^2(5x - 6) \end{aligned}$$

The critical numbers are given by $f'(x) = 0 \Leftrightarrow x^3(5x - 8) = 0 \Rightarrow x = 0$ or $x = \frac{8}{5}$, and since $f'(x)$ is a polynomial, no values of x will make $f'(x)$ undefined. In addition, no values of x will make $f''(x)$ undefined, and so the values of x which make $f''(x) = 0$ are $4x^2(5x - 6) = 0 \Rightarrow x = 0$ or $x = \frac{6}{5}$, which are possible inflection points. We use these values to find the intervals on which $f(x)$ is increasing, decreasing, concave upward or concave downward:

	0	8/5
x = −1	x = 1	x = 2
f '(−1) > 0	f '(1) < 0	f '(2) > 0
f ↑	f ↓	f ↑

	0	6/5
x = −1	x = 1	x = 2
f ''(−1) < 0	f ''(1) < 0	f ''(2) > 0
f is CD	f is CD	f is CU

Hence $f(x)$ is increasing on $(-\infty, 0)$ and $\left(\frac{8}{5}, \infty\right)$ and is decreasing on $\left(0, \frac{8}{5}\right)$. By the first derivative test, it follows that there is a relative maximum at $x = 0$, i.e. $(0, 0)$ and a relative minimum at $x = \frac{8}{5}$, i.e. $\left(\frac{8}{5}, \frac{-8192}{3125}\right)$. Furthermore, $f(x)$ is concave upward on $\left(\frac{6}{5}, \infty\right)$ and concave downward on $(-\infty, 0)$ and $\left(0, \frac{6}{5}\right)$. Since the concavity of f changes at $x = \frac{6}{5}$, it follows that $x = \frac{6}{5}$ is a point of inflection, and the coordinate is $\left(\frac{6}{5}, \frac{-5184}{3125}\right)$. For the x-intercepts, we set $f(x) = 0$:

$$f(x) = x^5 - 2x^4 = 0 \quad \Leftrightarrow \quad x^4(x - 2) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = 2$$

So the x-intercepts are: $(0, 0)$ and $(2, 0)$. This also gives the y-intercept as $y = 0$.

   (b) No, $(0, 0)$ is <u>not</u> an inflection point, since the concavity does not change there. In other words, the sign of $f''(x)$ does not change at $x = 0$.

2. $f(x) = x^3 + ax^2 + bx + c$.

We need to find the constants a, b, and c so that the function satisfies all the given information. Since $f(x)$ passes through the point $(0, 1)$, we have that

$$f(0) = (0)^3 + a(0)^2 + b(0) + c = 1 \quad \Rightarrow \quad c = 1$$

So the constant $c = 1$. Furthermore, since $x = 3$ is a critical number of $f(x)$, we know that $f'(3) = 0$ or is undefined. Since $f(x)$ is a polynomial, $f'(x)$ will also be a polynomial, which is defined for all real numbers. Therefore:

$$f'(x) = 3x^2 + 2ax + b \quad \text{so} \quad f'(3) = 3(3)^2 + 2a(3) + b = 27 + 6a + b = 0$$

This gives us an equation with a and b: $6a + b = -27$. Also, since $x = \frac{7}{6}$ is an inflection point, we know that $f''\left(\frac{7}{6}\right) = 0$ (it won't be undefined since $f''(x)$ will be a polynomial):

$$f''(x) = 6x + 2a \quad \text{so} \quad f''\left(\frac{7}{6}\right) = 6\left(\frac{7}{6}\right) + 2a = 7 + 2a = 0$$

This implies that $a = -\frac{7}{2}$. Since we have a, we can solve for b:

$$6\left(-\frac{7}{2}\right) + b = -27 \quad \Rightarrow \quad -21 + b = -27 \quad \Rightarrow \quad b = -6$$

So we have all three constants, and the polynomial of interest is

$$f(x) = x^3 - \frac{7}{2}x^2 - 6x + 1.$$

3. (a) $\displaystyle\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}=\lim_{x\to\infty}\frac{3\left(x-\frac{1}{6}\right)^2-\frac{25}{12}}{5\left(x+\frac{2}{5}\right)^2+\frac{1}{5}}=\frac{\infty}{\infty}$. Dividing the numerator and denominator by x^2 gives

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{3-\frac{1}{x}-\frac{2}{x^2}}{5+\frac{4}{x}+\frac{1}{x^2}}=\frac{3-0-0}{5+0+0}=\frac{3}{5}.$$

(b) Recall that $\lim_{x\to\infty}e^{-x}=0$ so $\displaystyle\lim_{x\to\infty}\frac{4}{2-5e^{-x}}=\frac{4}{2-0}=2.$

(c) $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}=\frac{\infty}{-\infty}$. Recall that, for $x<0$, $x=-\sqrt{x^2}$ so dividing the numerator and the denominator by x gives

$$\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{3x-5}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to-\infty}\frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{\frac{3x-5}{x}}=\lim_{x\to-\infty}\frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}}=-\frac{\sqrt{2+0}}{3-0}=-\frac{\sqrt{2}}{3}.$$

(d) $\displaystyle\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\infty-\infty$, which is an indeterminate form. But we want to have $\frac{\infty}{\infty}$, so let's rationalize the expression: $\displaystyle\left(\sqrt{x^2+1}-x\right)\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}=\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}=\frac{1}{\sqrt{x^2+1}+x}$. Further, recall that for $x>0$, $x=\sqrt{x^2}$. Then dividing the numerator and denominator by x gives

$$\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)=\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\sqrt{1+\frac{1}{x^2}}+1}=\frac{0}{\sqrt{1+0}+1}=\frac{0}{2}=0.$$

4. $\displaystyle f(x)=\frac{10x-10}{x^2-2x-8}$. For the x-intercepts, we set $f(x)=0$, so $10x-10=10(x-1)=0$, and hence $x=1$. For the y-intercept, we compute $f(0)=\frac{-10}{-8}=\frac{5}{4}$. So the x-intercept is $(1,0)$ and the y-intercept is $\left(0,\frac{5}{4}\right)$.

For the vertical asymptotes, we set $x^2-2x-8=(x-4)(x+2)=0$, so $x=4$ and $x=-2$. Both of these yield a nonzero numerator (determinate form):

$$\lim_{x\to4}\frac{10x-10}{x^2-2x-8}=\frac{30}{0}\quad\textbf{DF}\qquad\qquad\lim_{x\to-2}\frac{10x-10}{x^2-2x-8}=\frac{-30}{0}\quad\textbf{DF}$$

and so they are both vertical asymptotes. In fact, we see that

$$\lim_{x\to4^-}f(x)=\lim_{x\to-2^-}f(x)=-\infty\qquad\text{and}\qquad\lim_{x\to4^+}f(x)=\lim_{x\to-2^+}f(x)=\infty.$$

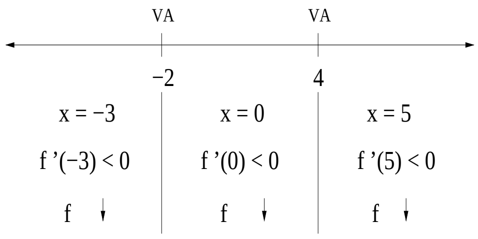
For the horizontal asymptotes, we have that $\displaystyle\lim_{x\to\infty}\frac{10x-10}{x^2-2x-8}=\lim_{x\to\infty}\frac{10x-10}{(x-1)^2-9}=\frac{\infty}{\infty}$, and so

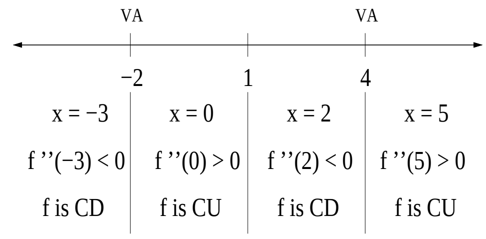
$$\lim_{x\to\infty}\frac{10x-10}{x^2-2x-8}\cdot\frac{\frac{1}{x^2}}{\frac{1}{x^2}}=\lim_{x\to\infty}\frac{\frac{10}{x}-\frac{10}{x^2}}{1-\frac{2}{x}-\frac{8}{x^2}}=\frac{0}{1}=0,$$

and similarly for $x\to-\infty$. Hence $y=0$ is the only horizontal asymptote.

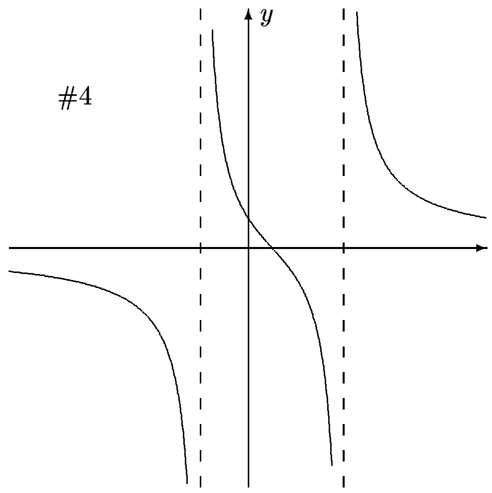
Next, we have $\displaystyle f'(x)=\frac{-10(x^2-2x+10)}{(x^2-2x-8)^2}$. If we set $f'(x)=0$ then $-10(x^2-2x+10)=0$ but this equation has imaginary roots, namely $1\pm i3$, so $f'(x)\neq0$ for any x. $f'(x)$ is undefined when $(x^2-2x-8)^2=0$ so $x=4$ or $x=-2$, which are the vertical asymptotes. Hence there are no critical points. From the sign pattern (using the points where f is undefined), we see that $f(x)$ is increasing nowhere and decreasing for $x<-2$, $-2<x<4$, and $x>4$.

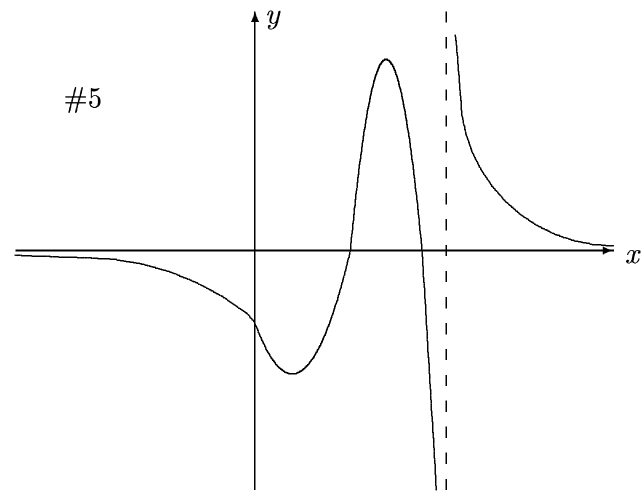
Finally, we have $\displaystyle f''(x)=\frac{20(x-1)(x^2-2x+28)}{(x^2-2x-8)^3}$. If we set $f''(x)=0$ then $20(x-1)(x^2-2x+28)=0$, so either $x=1$ or $x^2-2x+28=0$. This latter equation has imaginary roots $(1\pm i3\sqrt{3})$, however. Also, if $f''(x)$ is undefined then $(x^2-2x-8)^3=0$ so $x=4$ or $x=-2$ — again, these are just the vertical asymptotes. Hence the only possible





point of inflection is $x = 1$, and we see from the sign pattern that it is indeed one. We also note that $f(x)$ is concave up for $-2 < x < 1$ and $x > 4$, and concave down for $x < -2$ and $1 < x < 4$. The graph of $f(x)$ is given below.

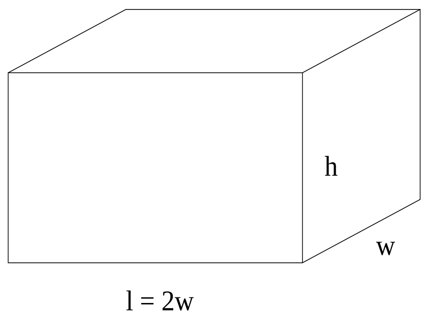


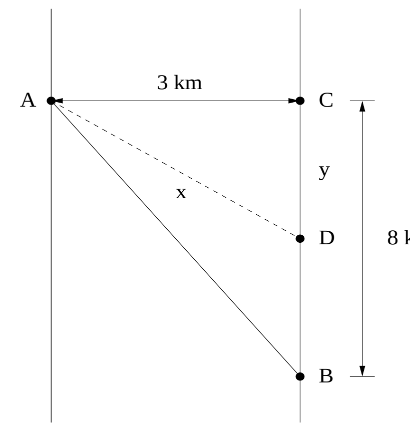


5. Interpret the information given: $f'(x) < 0$ means that the function f is decreasing on the intervals $(-\infty, 1)$, $(3, 4)$, and $(4, \infty)$. $f'(x) > 0$ means that f is increasing on the interval $(1, 3)$. f is decreasing followed by increasing at $x = 1$, so this is a relative minimum, and f is increasing followed by decreasing at $x = 3$, so this is a relative maximum point. $f''(x) < 0$ means that f is concave down on the intervals $(-\infty, 0)$ and $(2, 4)$. $f''(x) > 0$ means that f is concave up on $(0, 2)$ and $(4, \infty)$. Since the concavity of f changes at $x = 0$ and $x = 2$, and $f''(0) = f''(2) = 0$, then these are points of inflection. The concavity of f also changes at $x = 4$ but since $\lim_{x\to 4^-} f(x) = -\infty$ and $\lim_{x\to 4^+} f(x) = \infty$ it turns out that $x = 4$ is a vertical asymptote. Also, since $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, it follows that $y = 0$ is a horizontal asymptote (for both directions). Finally, $f(0) = -\frac{3}{2}$, $f(1) = -\frac{5}{2}$, $f(2) = 0$, $f(3) = 4$, $f\left(\frac{7}{2}\right) = 0$ gives the x– and y–intercepts and a point on the graph. The graph of the function which satisfies this information is given above.

6. Let w be the width, ℓ the length and h be the height of the box. The diagram labeled with these variables is shown below. The quantity to be minimized is the cost, C. The primary equation is
$$C = 10(w\ell) + 6[2(h\ell) + 2(wh)].$$
The secondary equations are $\ell = 2w$ and $w\ell h = 10$; combining these gives $h = \dfrac{5}{w^2}$. Then the primary equation becomes
$$C = C(w) = 10(2w^2) + 6\left[4w\left(\frac{5}{w^2}\right) + 2w\left(\frac{5}{w^2}\right)\right] = 20w^2 + \frac{180}{w}, \quad \text{Domain: } x > 0.$$
So $C'(w) = 40w - \dfrac{180}{w^2}$ which is zero when $40w = \dfrac{180}{w^2}$ or $w = \sqrt[3]{\frac{9}{2}}$. Since $C''(w) = 40 + \dfrac{360}{w^3}$, when $w = \sqrt[3]{\frac{9}{2}}$, $C'' > 0$ so by the Second Derivative Test we have a minimum. For $w = \sqrt[3]{\frac{9}{2}}$, $C = 20\left(\sqrt[3]{\frac{9}{2}}\right) + \frac{180}{\sqrt[3]{\frac{9}{2}}} \approx 163.54$, so the cheapest possible cost is \$163.54.

7. Let x be the distance the man rows and let y be the distance at which he lands from point C. The diagram with these variables appears above. We want to minimize the travel time, t. Since he will run $(8 - y)$ km, the primary equation is
$$t = \frac{x}{6} + \frac{8-y}{8},$$





obtained using the distance = speed × time formula. But y forms a right triangle with the distance from A to C, which is 3, and the hypotenuse is x so by Pythagoras, the secondary equation is $x^2 = 3^2 + y^2 \implies x = \sqrt{9+y^2}$. Then the primary equation becomes

$$t = \frac{1}{6}\sqrt{9+y^2} + \frac{8-y}{8}, \qquad \text{Domain: } 0 \le y \le 8.$$

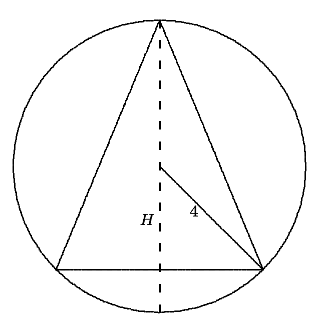
So $t' = \frac{1}{6}y(9+y^2)^{-\frac{1}{2}} - \frac{1}{8}$ and setting this to zero yields $y = \frac{9\sqrt{7}}{7}$. Note that $t'' = \frac{3}{2}(9+y^2)^{-\frac{3}{2}}$ which is always positive, so by the Second Derivative Test, this y does yield a minimum. Alternatively, we could use the method of finding absolute extrema on an interval by finding

$$\begin{aligned} t(y=0) &= \frac{1}{2} + 1 = \frac{3}{2}, \\ t(y=8) &= \frac{1}{6}\sqrt{73} + \frac{8-8}{8} = \frac{\sqrt{73}}{6} \approx 1.42, \\ t\left(\frac{9\sqrt{7}}{7}\right) &= \frac{1}{6}\sqrt{9+\frac{81}{7}} + 1 - \frac{9\sqrt{7}}{56} = \frac{2\sqrt{7}}{7} - \frac{9\sqrt{7}}{56} + 1 = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \end{aligned}$$

Comparing these values we see that $x = \frac{9\sqrt{7}}{7}$ does indeed yield a minimum. Thus the swimmer should land $\frac{9\sqrt{7}}{7} \approx 3.4$ km away from point C.

8. We draw the circle as depicted. Let b be its base and h be its height and we want to maximize its area, A. The primary equation is $A = \frac{1}{2}bh$.

Since the triangle is isosceles, its height must be measured along the diameter of the circle. Therefore, since the radius of the circle is 4, we can let $h = H + 4$. Then H forms a right triangle with a radius of the circle and one half of the triangle's base. Hence, by Pythagoras,



$$H^2 + \left(\frac{1}{2}b\right)^2 = 4^2 \implies b = 2\sqrt{16-H^2}.$$

This is our secondary equation. Substituting this into the primary equation, along with our equation for h, gives

$$A = \frac{1}{2}(2\sqrt{16-H^2})(H+4) = \sqrt{16-H^2}(4+H), \qquad \text{Domain: } 0 < H < 4.$$

Then $A' = \frac{-2(H+4)(H-2)}{\sqrt{16-H^2}}$ and setting this equal to zero yields $H = -4$ or $H = 2$. We can rule out $H = -4$ since then the triangle would have zero height, so the only possible extremum is $H = 2$. We have $A'' = \frac{-2(H^2-4H-8)}{\sqrt{16-H^2}(H-4)}$ so when $H = 2$, $A'' < 0$, and hence by the Second Derivative Test $H = 2$ does indeed yield a maximum. When $H = 2$, $h = 6$ and $b = 4\sqrt{3}$. So the largest possible isosceles triangle has height 6 and base $4\sqrt{3}$.