MEMORIAL UNIVERSITY OF NEWFOUNDLAND

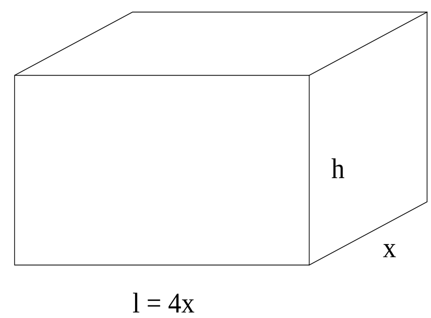
DEPARTMENT OF MATHEMATICS AND STATISTICS

HANDOUT **Mathematics 1000** FALL 2002

Applied Maximum and Minimum Problems

1. A rectangular storage container with a closed top is to have a volume of 32 m^3. The length of the base is 4 times the width. Material for the base and top costs \$8 per square metre, and material for the sides is \$6 per square metre. Find the cost of materials for the cheapest such container.

<u>**Solution:**</u> Let x be the width, ℓ the length and h be the height of the box, as shown in the diagram. Since the length of the base is four times the width, we know that $\ell = 4x$. We want to minimize the cost of the container, which is found using the surface area of the box. Hence the primary equation is a cost function, C, which is a function of x and h.



$$\begin{aligned} C &= (\$8\,/\mathrm{m}^2(\underbrace{\text{area of the base and top}}_{\text{units of m}^2}) + (\$6\,/\mathrm{m}^2(\underbrace{\text{area of the four sides}}_{\text{units of m}^2}) \\ &= 8(\underbrace{4x^2}_{\text{base}} + \underbrace{4x^2}_{\text{top}}) + 6(\underbrace{4xh + 4xh + xh + xh}_{\text{sides}}) \\ &= 64x^2 + 60xh. \end{aligned}$$

We want to express this as a function of a single variable (x or h), and so we use the secondary equation using the known volume of the box:

$$V = 4x \cdot x \cdot h \quad \Rightarrow \quad 32 = 4x^2 h \quad \Rightarrow \quad h = \frac{8}{x^2}$$

Substituting this value gives a new primary equation:

$$C = 64x^2 + 60x\left(\frac{8}{x^2}\right) = 64x^2 + \frac{480}{x} = 8\left[8x^2 + \frac{60}{x}\right] \qquad \text{Domain: } x > 0.$$

Next we differentiate with respect to the lone independent variable x, and find the critical numbers:

$$C'(x) = 8\left[16x - \frac{60}{x^2}\right] = 8\left[\frac{16x^3 - 60}{x^2}\right] = 32\left[\frac{4x^3 - 15}{x^2}\right]$$

$$C'(x) = 0 \quad \Leftrightarrow \quad 32\left[\frac{4x^3 - 15}{x^2}\right] = 0 \quad \Rightarrow \quad 4x^3 = 15 \quad \Rightarrow \quad x = \sqrt[3]{\frac{15}{4}}\ \mathrm{m}$$

Therefore the width of the box should be $x = \sqrt[3]{\frac{15}{4}}$ m and the length should be $4x = 4\sqrt[3]{\frac{15}{4}}$ m, and the height should be $h = \frac{8}{x^2} = \frac{8}{\left(\frac{15}{4}\right)^{\frac{1}{3}\cdot 2}} = 8\left(\frac{4}{15}\right)^{\frac{2}{3}}$ m. We must verify this gives a minimum value, and so we use the Second Derivative Test:

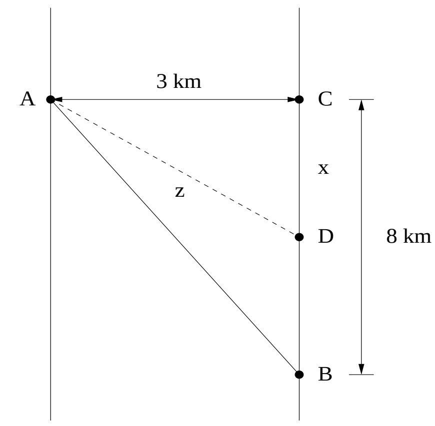
$$C''(x) = 64\left[\frac{2x^3 + 15}{x^3}\right] \quad \Rightarrow \quad C''\left(\sqrt[3]{\frac{15}{4}}\right) = 64(6) = 384 > 0$$

and so this value of x is a <u>relative minimum</u>. The cost of the container with these dimensions is

$$C\left(\sqrt[3]{\frac{15}{4}}\right) = 64\left(\sqrt[3]{\frac{15}{4}}\right)^2 + \frac{480}{\left(\sqrt[3]{\frac{15}{4}}\right)} \approx \$463.43.$$

2. There is a power source at point A on the bank of a straight river 3 km wide and a city at point B 8 km downstream on the opposite bank. We want to lay cable connecting A and B as cheaply as possible , and it costs \$ 12 per km to lay cable under water and only \$ 3 per km to lay cable over land. The route which requires the least amount of underwater cable is the route directly across the river A to C and then overland from C to B, but this would require the most cable. In general, the cable could be laid from A to a point D on the opposite bank between C and B and then from D to B. Where should this point D be in order to minimize the total cost?

<u>**Solution:**</u> Let x = distance from point C to point D (in km), and z = distance across the river from point A to point D. Since the distance from C to B is given as 8 km, then the distance from D to B must be $8-x$ km. The figure depicting this situation is given below. The line connecting A to D is the hypotenuse of a right triangle with sides 3 km (the width of the river) and x. So we use Pythagoras to find the distance from A to D:



$$z^2 = x^2 + 3^2 \quad \Rightarrow \quad z = \sqrt{x^2+9}$$

For the **primary equation**, we want to minimize the cost of laying cable from A to D to B, and so the total cost will be the distance from A to D in km times the cost of laying cable per km underwater plus the distance from D to B times the cost of laying cable per km over land. So the primary equation is a cost function, C:

$$\begin{aligned} C &= (\$12\,/\text{km})\left(\sqrt{x^2+9}\,\text{km}\right) + (\$3\,/\text{km})(8 - x\,\text{km}) \\ C(x) &= 12\sqrt{x^2+9} + 3(8-x) \qquad \text{Domain: } 0 \le x \le 8. \end{aligned}$$

To minimize the cost function, we differentiate and find the critical numbers:

$$C'(x) = \frac{12x}{\sqrt{x^2+9}} - 3 = 0 \quad \Rightarrow \quad \frac{12x}{\sqrt{x^2+9}} = 3 \quad \Rightarrow \quad 144x^2 = 9(x^2+9) \quad \Rightarrow \quad x^2 = \frac{3}{5}$$

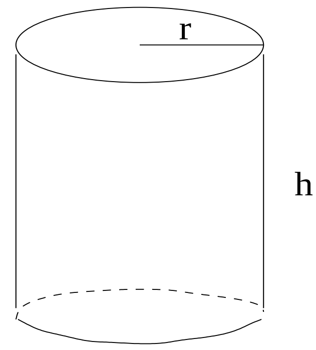
which gives the critical numbers $x = \pm\sqrt{\frac{3}{5}} = \pm\frac{\sqrt{15}}{5}$ after rationalizing. We can reject the negative solution since $0 \le x \le 8$ and so x cannot be negative. Now we must verify that this value of x gives a minimum value of the cost function. We can do this by using the method of finding absolute extrema on a closed interval:

$$\begin{aligned} C\left(\frac{\sqrt{15}}{5}\right) &= 12\sqrt{\frac{3}{5}+9} + 3\left(8 - \frac{\sqrt{15}}{5}\right) \approx \$58.86 \qquad \text{Route: A} \to \text{D} \to \text{B} \\ C(0) &= 12\sqrt{0+9} + 3000(8-0) = \$60 \qquad \text{Route: A} \to \text{C} \to \text{B} \\ C(8) &= 12\sqrt{8^2+9} + 3(8-8) = 12\sqrt{73} \approx \$102.53 \qquad \text{Route: A} \to \text{B} \end{aligned}$$

Therefore the cost is minimized by using the route A $\to$ D $\to$ B; and $x = \frac{\sqrt{15}}{5}$ is both a relative and absolute minimum. Hence the point D should be $\frac{\sqrt{15}}{5}$ km from the point C in order to minimize the cost.

3. A Pepsi can holds 355 ml = 0.355 L of liquid, and has the shape of a right circular cylinder. Find the dimensions of the can that will minimize the total cost of materials to make the can. Hint: 1 L = 1000 cm^3.

Solution: Let r be the radius of the circular top of the can, and h be the height of the can, both in cm, as shown in the figure. To minimize the total cost of the can, we need to minimize the total surface area of the cylinder (top, bottom, and side). And so the **primary equation** is



$$A = \underbrace{\pi r^2}_{\text{Area of top}} + \underbrace{\pi r^2}_{\text{Area of bottom}} + \underbrace{2\pi r h}_{\text{Area of the rectangle}} = 2\pi r^2 + 2\pi r h$$

Note that the area of the "side" of the cylinder (i.e. not the circular top or bottom) is simply the area of the rectangle with width h and length the circumference of the circle, $2\pi r$. We want a primary equation in terms of only one variable, and so to eliminate h in terms of r we construct a secondary equation from the volume information $V = 355\,\text{mL} = \frac{355}{1000}\,\text{L} = 355\,\text{cm}^3$:

$$V = \pi r^2 h \quad \Rightarrow \quad 355 = \pi r h^2 \quad \Rightarrow \quad h = \frac{355}{\pi r^2}$$

Substituting this into the primary equation gives a function of r only, $A = A(r)$

$$A(r) = 2\pi r^2 + 2\pi r\left(\frac{355}{\pi r^2}\right) = 2\pi r^2 + \frac{710}{r}, \qquad \text{Domain: } r > 0$$

Differentiate and find the critical numbers:

$$A'(r) = 4\pi r - \frac{710}{r^2} = \frac{2(2\pi r^3 - 355)}{r^2} = 0 \quad \Rightarrow \quad 2\pi r^3 - 355 = 0 \quad \Rightarrow \quad r = \sqrt[3]{\frac{355}{2\pi}}\ \text{cm}$$

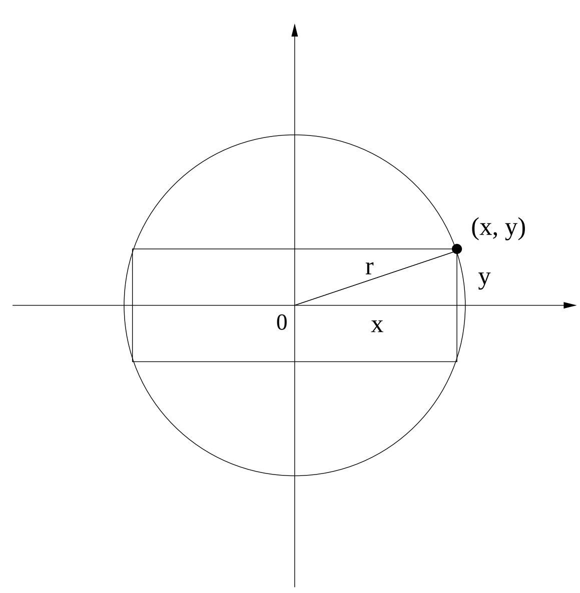
We use the second derivative test to confirm that $A(r)$ is minimized when $r = \sqrt[3]{\frac{355}{2\pi}}$

$$A''(r) = 4\pi + \frac{1420}{r^3} \quad \Rightarrow \quad A''\left(\sqrt[3]{\frac{355}{2\pi}}\right) = 4\pi + 8\pi = 12\pi > 0$$

and hence does yield a minimum surface area. The height and radius are related by $h = \frac{355}{\pi r^2}$, and at this value of r the height is $h = \frac{355}{\pi\left(\frac{355}{2\pi}\right)^{\frac{2}{3}}} = \frac{355^{\frac{1}{3}} \cdot 4^{\frac{1}{3}}}{\pi^{\frac{1}{3}}} = \sqrt[3]{\frac{1420}{\pi}} = 2\sqrt[3]{\frac{355}{2\pi}}$ cm. There for to minimize the cost of the can, the radius should be $\sqrt[3]{\frac{355}{2\pi}}$, and the height should be twice the radius.

4. Find the area of the largest rectangle that can be inscribed in a circle of radius r (general radius).

Solution: To say that a rectangle is inscribed in a circle means that its vertices are on the circumference of the circle. To make this problem easier, we place the circle and rectangle on a coordinate axes so that the center of the circle coincides with the origin, as depicted in the figure. Let (x, y) be the coordinates of the vertex of the rectangle in the first quadrant, and so the line connecting the origin to (x, y) has length r, the general radius of the circle. We want to maximize the area of the rectangle, which, in this setup, has dimensions length$= 2x$ and width$= 2y$, and so the **primary equation** of area is



$$A = 2x \cdot 2y = 4xy$$

We want to express this as a function of a single variable, say x. To do so we take as the secondary equation the equation of the circle $x^2 + y^2 = r^2$. We know x and y are related like this because (x, y) is a point on the circumference of the circle. Solving for y we obtain $y = \sqrt{r^2 - x^2}$, and substituting this into the primary equation we get

$$A(x) = 4x\sqrt{r^2 - x^2} \qquad \text{Domain: } r^2 - x^2 \geq 0 \quad \Rightarrow \quad (r + x)(r - x) \geq 0 \quad \Rightarrow \quad 0 \leq x \leq r,$$

which is a function of a single variable. Differentiate to find the critical values

$$A'(x) = 4\sqrt{r^2 - x^2} - \frac{4x^2}{\sqrt{r^2 - x^2}} = \frac{4(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \quad \Rightarrow \quad 4(r^2 - x^2) = 0 \quad \Rightarrow \quad x^2 = \frac{r^2}{2}$$

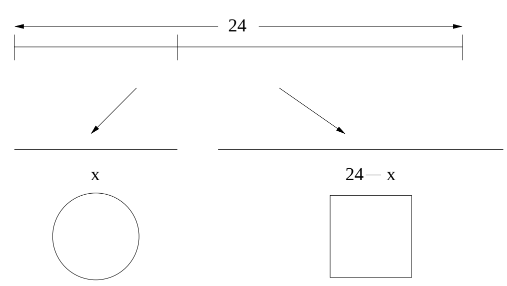
which yields the critical numbers $x = \frac{r}{\sqrt{2}} = \frac{r\sqrt{2}}{2}$, where we reject the negative solution since we must have $x \geq 0$. Since we have a nice, well–defined domain for our primary equation, we can use the method of finding absolute extrema on a closed interval to verify that this value does give a maximum area. So we evaluate the area function at the critical points and endpoints:

$$A(0) = 0, \quad A(r) = 0, \quad A\left(\frac{r\sqrt{2}}{2}\right) = 4\left(\frac{r\sqrt{2}}{2}\right)\sqrt{r^2 - \frac{2r^2}{4}} = 2\sqrt{2}r\sqrt{\frac{r^2}{2}} = 2r^2,$$

and so the maximum area is indeed obtained when $x = \frac{r\sqrt{2}}{2}$. Hence the largest rectangle that can be inscribed in a circle of radius r has length $r\sqrt{2}$, width $r\sqrt{2}$ and area $A = 2r^2$.

5. A piece of wire 24 cm long is to be cut into 2 pieces, one bent into a circle, the other into a square. How should the wire be cut so as to minimize the total area enclosed by the figures?

Solution: Let x be the length of wire used to form the circle. Therefore, the wire used to form the square must have length $24-x$. Let r be the radius of the circle and ℓ the length of the sides of the square. We want to minimize the total area of the circle and square. Hence the **primary equation** is



$$A = A_{\circ} + A_{\square} = \pi r^2 + \ell^2$$

Note that x is the circumference of the circle, and so $x = 2\pi r \Rightarrow r = \frac{x}{2\pi}$ is the radius of the circle. Since the perimeter of the square is $24 - x$, the the length of each side of the square is $\ell = \frac{24-x}{4}$. With these secondary equations, we obtain the primary equation as a function of a single variable x

$$A = \pi r^2 + \ell^2 = \pi\left(\frac{x}{2\pi}\right)^2 + \left(\frac{24 - x}{4}\right)^2 = \frac{x^2}{4\pi} + \left(6 - \frac{x}{4}\right)^2 \qquad \text{Domain: } 0 \leq x \leq 24$$

$$A(x) = \frac{x^2}{4\pi} + 36 - 3x + \frac{x^2}{16} = \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - 3x + 36 = \left(\frac{4 + \pi}{16\pi}\right)x^2 - 3x + 36$$

Differentiate and find the critical numbers:

$$A'(x) = \left(\frac{4 + \pi}{16\pi}\right)2x - 3 = \left(\frac{4 + \pi}{8\pi}\right)x - 3 = 0 \quad \Rightarrow \quad x = 3\left(\frac{8\pi}{4 + \pi}\right) = \frac{24\pi}{4 + \pi}\text{ cm}$$

Now we verify that $A(x)$ is a minimum at x, using the second derivative test:

$$A''(x) = \frac{4 + \pi}{8\pi} \quad \Rightarrow \quad A''\left(\frac{24\pi}{4 + \pi}\right) > 0 \quad \Rightarrow \quad A \text{ is a minimum at } x = \frac{24\pi}{4 + \pi}$$

Therefore to minimize the total area of the figures, we should cut the wire so that the piece used to form the circle has length $\frac{24\pi}{4+\pi}$ cm.