Differentiation Worksheet

Solutions

1. Find the derivative of the following functions, using the appropriate differentiation rules.

(a) $y = \left(x + \sqrt{\ln(x)}\right)^3 = \left(x + (\ln(x))^{\frac{1}{2}}\right)^3$

$$\frac{dy}{dx} = 3\left(x + (\ln(x))^{\frac{1}{2}}\right)^2 \left[1 + \frac{1}{2}(\ln(x))^{-\frac{1}{2}}\left(\frac{1}{x}\right)\right] = 3\left(x + \sqrt{\ln(x)}\right)^2 \left[1 + \frac{1}{2x\sqrt{\ln(x)}}\right]$$

(b) $f(x) = \sin^3(1-3x)^5 = (\sin(1-3x)^5)^3$

$$\begin{aligned} f'(x) &= 3(\sin(1-3x)^5)^2 \left[\cos(1-3x)^5 \cdot 5(1-3x)^4(-3)\right] \\ &= -45(1-3x)^4 \sin^2(1-3x)^5 \cos(1-3x)^5 \end{aligned}$$

(c) $g(t) = \tan^2(\sec(\sin\left(t^3\right))) = \left[\tan(\sec(\sin\left(t^3\right)))\right]^2$

$$\begin{aligned} g'(t) &= 2\tan(\sec(\sin\left(t^3\right))) \cdot \sec^2(\sec(\sin\left(t^3\right))) \cdot \sec(\sin\left(t^3\right))\tan(\sin\left(t^3\right)) \cdot \cos\left(t^3\right) \cdot 3t^2 \\ &= 6t^2 \tan(\sec(\sin\left(t^3\right))) \sec^2(\sec(\sin\left(t^3\right))) \sec(\sin\left(t^3\right)) \tan(\sin\left(t^3\right)) \cos\left(t^3\right) \end{aligned}$$

(d) $y = e^{x\cos(x)}$

$$y' = e^{x\cos(x)}\left[(1)\cos(x) + (-\sin(x))x\right] = (\cos(x) - x\sin(x))\, e^{x\cos(x)}$$

(e) $y = \left(x^4 + \sqrt{x + \sqrt[3]{x-1}}\right)^{\sqrt{2}}$

$$\begin{aligned} \frac{dy}{dx} &= \sqrt{2}\left(x^4 + \sqrt{x + \sqrt[3]{x-1}}\right)^{\sqrt{2}-1} \frac{d}{dx}\left[x^4 + \left(x + (x-1)^{\frac{1}{3}}\right)^{\frac{1}{2}}\right] \\ &= \sqrt{2}\left(x^4 + \sqrt{x + \sqrt[3]{x-1}}\right)^{\sqrt{2}-1} \left[4x^3 + \frac{1}{2}\left(x + (x+1)^{\frac{1}{3}}\right)^{-\frac{1}{2}} \left(1 + \frac{1}{3}(x-1)^{-\frac{2}{3}}(1)\right)\right] \\ &= \sqrt{2}\left(x^4 + \sqrt{x + \sqrt[3]{x-1}}\right)^{\sqrt{2}-1} \left[4x^3 + \frac{1}{2\sqrt{x + \sqrt[3]{x-1}}}\left(1 + \frac{1}{3(x-1)^{\frac{2}{3}}}\right)\right] \end{aligned}$$

(f) $g(s) = \sin(\cos(\tan(s)))$

$$\begin{aligned} g'(s) &= \cos(\cos(\tan(s)))\frac{d}{ds}\left[\cos(\tan(s))\right] \\ &= \cos(\cos(\tan(s)))\left[-\sin(\tan(s))\right]\frac{d}{ds}\left[\tan(s)\right] \\ &= -\cos(\cos(\tan(s)))\sin(\tan(s))\sec^2(s) \end{aligned}$$

2. Find the derivative of the following functions using the <u>definition of derivative</u>.

(a) $f(x) = 2x^2 - 3x + 6$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2(x+h)^2 - 3(x+h) + 6 - (2x^2 - 3x + 6)}{h} \\ &= \lim_{h\to 0} \frac{2x^2 + 4hx + 2h^2 - 3x - 3h + 6 - 2x^2 + 3x - 6}{h} \\ &= \lim_{h\to 0} \frac{4hx + 2h^2 - 3h}{h} \\ &= \lim_{h\to 0} \frac{h(4x + 2h - 3)}{h} = 4x - 3 \end{aligned}$$

(b) $g(x) = \sqrt{2-x}$

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2-(x+h)}-\sqrt{2-x}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{2-x-h}-\sqrt{2-x}}{h}\left(\frac{\sqrt{2-x-h}+\sqrt{2-x}}{\sqrt{2-x-h}+\sqrt{2-x}}\right) \\ &= \lim_{h\to 0} \frac{2-x-h-(2-x)}{h(\sqrt{2-x-h}+\sqrt{2-x})} \\ &= \lim_{h\to 0} \frac{-h}{h(\sqrt{2-x-h}+\sqrt{2-x})} = \frac{-1}{2\sqrt{2-x}} \end{aligned}$$

(c) $f(t) = \dfrac{3t-1}{2t-1}$

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \\ &= \lim_{h\to 0} \frac{\frac{3(t+h)-1}{2(t+h)-1} - \frac{3t-1}{2t-1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{(3t+3h-1)(2t-1)-(3t-1)(2t+2h-1)}{(2t+2h-1)(2t-1)}}{h} \\ &= \lim_{h\to 0} \frac{6t^2-5t+6ht-3h+1-(6t^2+6ht-5t-2h+1)}{h(2t+2h-1)(2t-1)} \\ &= \lim_{h\to 0} \frac{-h}{h(2t+2h-1)(2t-1)} = \frac{-1}{(2t-1)^2} \end{aligned}$$

(d) $h(x) = \dfrac{1}{x^2-x-2}$

$$\begin{aligned} h'(x) &= \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2-(x+h)-2} - \frac{1}{x^2-x-2}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x^2-x-2-((x+h)^2-(x+h)-2)}{((x+h)^2-(x+h)-2)(x^2-x-2)}}{h} \\ &= \lim_{h\to 0} \frac{x^2-x-2-x^2-2hx-h^2+x+h+2}{h((x+h)^2-(x+h)-2)(x^2-x-2)} \\ &= \lim_{h\to 0} \frac{-2hx-h^2+h}{h((x+h)^2-(x+h)-2)(x^2-x-2)} \\ &= \lim_{h\to 0} \frac{-2x-h+1}{((x+h)^2-(x+h)-2)(x^2-x-2)} = \frac{-2x+1}{(x^2-x-2)^2} \end{aligned}$$

3. **Proofs**: See class notes for proofs of the following using the definition of derivative:

- The Product Rule $\dfrac{d}{dx}[f(x)g(x)] = f'(x)g(x) + g'(x)f(x)$
- The Quotient Rule $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$
- The Chain Rule $\dfrac{d}{dx}[f(g(x)] = f'(g(x))g'(x)$
- The Constant Multiple Rule $\dfrac{d}{dx}[kf(x)] = kf'(x)$
- The Sum/ Difference Rule $\dfrac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
- Trigonometric derivatives $\dfrac{d}{dx}[\sin(x)] = \cos(x)$, and $\dfrac{d}{dx}[\cos(x)] = -\sin(x)$