Department of Mathematics and Statistics

Acadia University
Wolfville, Nova Scotia
Canada B4P 2R6

Telephone: (902) 585-1862
Facsimile: (902) 585-1074
Email: ronald.haynes@acadiau.ca

June 12, 2007

Reaching Out Vol. 1, No. 3
Mathematics Students
Lewisporte Collegiate

To whom it may concern:

Hopefully if you are reading this it means that you enjoyed the first two installments of this newsletter and have come back for more mathematics. Just a quick re–introduction if this is your first installment of the newsletter. I was a student at Lewisporte Regional High School and a graduate of 1992. I attended Memorial University and studied mathematics and then went on to Simon Fraser University in British Columbia and completed both a MSc and PhD in applied and computational mathematics. Since July 2004 I have been a faculty member in the department of mathematics and statistics at Acadia University in Nova Scotia.

In this installment I will continue our discussion of sequences and series. Please refer to my website (listed in the next paragraph) to download previous issues. In the previous issue you were introduced to the various representations of sequences and series as well as some details regarding finite arithmetic and geometric sequences. Here we will look at infinite sequences and series and discuss issues regarding convergence. At the end of this letter you will hints and solutions for the problems from the previous installment as well as a new set of problems to keep you busy over the summer.

Please do not hesitate to contact me at *ronald.haynes@acadiau.ca* should you have questions. You can also visit my website *http://rhaynes.acadiau.ca/personal* to find out more info about my work and teaching as well as links to our department and Acadia. Links to previous installments of this newsletter may be found at this website as well.

Sequences and Series Part II

As we saw last time a sequence is simply a list of numbers with order. For example, the following is an infinite geometric sequence:

$$\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}.$$

The three dots indicate that the sequence and any pattern established in the first few terms continues forever. To refer to particular terms in a sequence we often use a subscript system. We will use t_n to denote the nth term in the sequence. So in the example above the third term, t_3 is $\frac{1}{4}$. The terms in a sequence do not necessarily follow a pattern but often they do. It is easy to come up with a formula for a general term in the above sequence, in fact $t_n = \left(\frac{1}{2}\right)^{n-1}$ for $n = 1, 2, \ldots$. In higher mathematics we spend quite a bit of time trying to understand if infinite sequences converge. Essentially what we are trying to do is to determine if there exists a (finite) number L such that the terms in the sequence t_n get closer and closer to L as n gets larger and larger. If such a number

L exists then we call L the limit of sequence. In the example above we can make the terms in the sequence as close as we wish to the number $L = 0$ by taking n larger and larger. Suppose we want the terms to be within $1/2^{32}$ of the limit. This will occur from the 33^{rd} term of the sequence. In general, we can select any distance $\epsilon > 0$ and deduce that for $n > 1 + \frac{\ln \epsilon}{\ln 2}$ then t_n will be within ϵ of $L = 0$. The concept of a limit will be made formal in a course known as *real analysis* which math students normally take in second or third year.

If we attempt to add up the terms of an infinite sequence then we will arrive at an infinite series, for example

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

is an infinite geometric series. What does it mean to add up an infinite list of numbers? Clearly this is not possible to actually do, however, we can assign meaning to this sort of sum in some instances. Suppose you add up the first 5 terms of the above series you will get a sum of 1.9375, adding the first 10 terms gives 1.9980 and adding the first 20 terms gives a sum of approximately 1.99999809265137. It appears that as you add more and more terms the sums (called partial sums) seem to be approaching 2. When the partial sums converge then we say we can add up an infinite list of numbers or that the infinite series converges.

Infinite geometric series are relatively easy to study. A general infinite geometric series may be written as

$$\sum_{n=0}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots ,$$

where r is assumed to be not 1. In this case the sum of the first n terms, s_n is simply the sum of the finite geometric series

$$s_n = a + ar + ar^2 + \cdots ar^{n-1}.$$

If we multiply the formula for s_n by r we obtain

$$rs_n = ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n.$$

Subtracting these two equations we find

$$(1 - r)s_n = a(1 - r^n),$$

or

$$s_n = \frac{a(1 - r^n)}{1 - r}.$$

Now what happens to s_n as n gets large? The only quantity on the right hand side is r^n. Now if $r \neq 1$ then are three cases. If $|r| > 1$ then r^n gets arbitrarily large as n gets large which means that s_n will not converge and hence the infinite series does not converge – it diverges. If $r = -1$ then r^n simply cycles between 1 and -1 and again we don't have convergence. Convergence is only obtained if $|r| < 1$, then r^n converges to 0 and s_n converges to $a/(1-r)$. To summarize, the infinite geometric series converges only if $|r| < 1$ and in this case the sum is $a/(1 - r)$. The series above has $a = 1$ and $r = 1/2$, so $a/(1 - r) = 1/(1 - 1/2) = 2$.

Here are some problems to keep you busy over the summer. The hints and solutions for last months newsletter can be found after the new problems.

Problems for Vol. 1 No. 3

Problem 1. Consider the sequence with general term a_n and the sum of the first n terms given by $s_n = 2 + n/(3n+1)$. Determine s_1, s_2 and s_3. Determine a_1, a_2 and a_3. Determine a general formula for a_n. What is the sum of the infinite series $a_1 + a_2 + \cdots$.

Problem 2. The sum to infinity of a geometric series is 8, whereas the sum of the second and third terms is 3. Determine all possible values of the common ratio r.

Problem 3. Do some research to understand what $n!$ means (this is read as n factorial). Consider the sequence

$$a_n = \frac{3^n}{n!}.$$

Does this sequence converge? If so, to what?

Hints and Solutions for Vol. 1 No. 2

Problem 1. The sum of $200 - 199 + 198 - 197 + 196 - \cdots + 2 - 1$ is what?

The basic idea here is to group the sum as $(200 - 199) + (198 - 197) + (196 - 195) + \cdots + (2 - 1)$. Each quantity in the brackets is 1 and hence the sum is 100. Note: it is always possible to regroup finite sums, the same is not always true for infinite sums.

Problem 2. The sum of the first n terms of a sequence is $n(n+1)(n+2)$. What is the 10th term of the sequence?

If we let s_n denote the sum of the first n terms then we can obtain t_n by computing $s_n - s_{n-1}$ (WHY?). Therefore $t_{10} = s_{10} - s_9 = 10(11)(12) - 9(10)(11)$.

Problem 3. The three positive numbers a, b and c form a geometric sequence. Prove that the numbers $\log a$, $\log b$ and $\log c$ form an arithmetic sequence.

This is harder! If a, b, and c form a geometric sequence this means that there exists a number r such that $b = ar$ and $c = ar^2$. Therefore, $\log b = \log(ar) = \log a + \log r$. So $\log b$ is $\log r$ larger than $\log a$. In the same manner $\log c = \log(ar^2) = \log a + 2\log r$ (WHY?), which says that $\log c = \log b + \log r$ and we are done.

Problem 4. If a, b and c form an arithmetic sequence, show that

$$(b-c)x^2 + (c-a)x + (a-b) = 0$$

has equal roots.

A quadratic will have equal roots if the discriminant equals zero. If a, b and c form an arithmetic sequence then $b = a + d$ and $c = a + 2d$. Therefore $b - c = -d$, $c - a = 2d$ and $a - b = -d$ and the discriminant is given by

$$(c-a)^2 - 4(b-c)(a-b) = (2d)^2 - 4(-d)(-d) = 4d^2 - 4d^2 = 0.$$

Problem 5. For a geometric sequence of positive terms to have any term equal to the sum of the next two terms, what must the common ratio be?

Suppose the three terms are given by a, ar, ar^2 where $a, r > 0$. Now we want $a = ar + ar^2$ or $a - ar - ar^2 = 0$ Since $a \neq 0$ (by assumption) then we can divide through by a to get the quadratic equation $1 - r - r^2 = 0$ for r. Solving this equation with the quadratic formula we have

$$r = \frac{1 \pm \sqrt{5}}{2}.$$

The positive root is the desired r.

Sincerely,

Dr. Ronald D. Haynes