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Mathematics Students
Lewisporte Collegiate

To whom it may concern:

I would like to take this opportunity to welcome you to my (hopefully) regular mathematical newsletter. I hope through this correspondence to meet some students at Lewisporte Collegiate who are interested in mathematics. I was a student at Lewisporte Regional High School (same site) and a graduate of 1992 (jeez that sounds like a long time ago!). I attended Memorial University and studied mathematics and then went on to Simon Fraser University in British Columbia and completed both a MSc and PhD in applied and computational mathematics. Since July 2004 I have been a faculty member in the department of mathematics and statistics at Acadia University in Nova Scotia.

Below I have included some info on finding roots of higher order polynomials as well as some neat problems involving functions of roots of polynomials. I hope you find it interesting.

Please do not hesitate to contact me at *ronald.haynes@acadiau.ca* should you have questions. You can also visit my website *http://rhaynes.acadiau.ca/personal* to find out more info about my work and teaching as well as links to our department and Acadia.

Roots of Polynomials

One of the basic tasks presented to high school algebra students is to locate the roots of polynomials. We denote a polynomial $P(x)$ as

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where the numbers $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients and n is the degree of the polynomial. A number r is a root of $P(x)$ if $P(r) = 0$.

If $n = 1$ and $a_1 \neq 0$ then the polynomial is linear and there is at most one root

$$r = \frac{a_0}{a_1}, \quad \text{provided } a_1 \neq 0.$$

Early on students are presented with the quadratic formula which provides the roots of any second degree ($n = 2$, $a_2 \neq 0$) polynomial:

$$r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2 a_0}}{2a_2}.$$

There are special techniques which work in some instances for cubic (degree 3) and quartic (degree 4) polynomials. I will consider in some detail *Cardan's or del Ferro's method* for cubics. I will leave some of the steps for you to verify.

Let $P(x) = x^3 + ax^2 + bx + c$ (with $a \neq 0$) be a general cubic polynomial. (Why can I assume the coefficient of x^3 is one for purposes of finding the roots?) Now let $x = t + \alpha$ and find an expression for $P(t)$. What choice of α reduces this to a polynomial of the form $P(t) = t^3 + pt + q$? With this value of α what the expressions for p and q? Now let $t = u + v$, this gives (verify this) the polynomial in u and v:

$$(u^3 + v^3) + (3uv + p)(u + v) + q = 0.$$

Now this equation will be satisfied if $(u^3 + v^3) = -q$ and $3uv + p = 0$. So $u^3 + v^3 = -q$ and $u^3v^3 = -p^3/27$. Hence u^3 and v^3 are roots of the quadratic equation

$$z^2 + qz - p^3/27 = 0$$

which we can solve with usual techniques. Once we have u^3 and v^3 we can find u and v, then t and finally x.

A similar strategy exists for quartic polynomials as well. Tackling higher degree polynomials is more of a struggle. One very simple idea is that of *detection* and *reduction*. That is, we first find *a* root and then use that root to reduce the degree of the polynomial. The basic tool here is the *rational roots theorem* which says that if r is a rational root of $P(x)$ then the numerator of r must be an integer factor of a_0 and the denominator of r must be an integer factor of a_n. This provides us with a list of possible roots r_i which we can check by simply computing $P(r_i)$. If $P(r_i) = 0$ then r_i is a root of $P(x)$ which means that $x - r_i$ is a factor of $P(x)$. Hence we divide $P(x)$ by $x - r_i$ to obtain the remaining factor. This ultimately reduces the degree of the polynomial to be factored by 1. If the original polynomial is a cubic and we find one root and divide then we are left with a quadratic to solve. This process can be repeated recursively to find roots of high degree polynomials. Its success, however, requires that the original polynomial has a sufficient number of rational roots.

Another popular class of problems have us to deduce relationships amongst the roots of polynomials without actually finding the roots themselves. For example, if r_1 and r_2 are roots of x^2+bx+c then the sum of roots is $-b$ and the product of the roots is c. To see this simply expand

$$(x - r_1)(x - r_2)$$

and compare to $x^2 + bx + c$. If the quadratic is given by $ax^2 + bx + c$ then $r_1 + r_2 = -b/a$ and $r_1r_2 = c/a$ (Can you see why?). Using the relations $r_1 + r_2 = -b$ and $r_1r_2 = c$ we can find values for other quantities. Suppose we wish to find the sum of the squares of the roots, $r_1^2 + r_2^2$. Then we can proceed as follows

$$\begin{aligned} r_1^2 + r_2^2 &= (r_1 + r_2)^2 - 2r_1r_2 \\ &= (-b)^2 - 2c \\ &= b^2 - 2c. \end{aligned}$$

Below are a few problems based on the material above. I will give some hints and solutions in the next month in Issue 2 of the newsletter.

Problem 1. If r and s are roots of $ax^2 + bx + c = 0$ then find a value of

$$\frac{1}{r^2} + \frac{1}{s^2}$$

in terms of a, b and c.

Problem 2. If r and $2r$ are roots of $ax^2+bx+c=0$ then a, b and c are related by what expression?

Problem 3. If r and s are roots of x^2+x+7 then compute the value of

$$2r^2+rs+s^2+r+7$$

without actually finding r and s.

Sincerely,

Dr. Ronald D. Haynes