1 If the Wronskin of f and g is $3e^{4t}$, and if $f(t)=e^{2t}$. find $g(t)$

Solution:

$$W=\begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = e^{2t}g'(t) - g(t)\cdot 2e^{2t} = 3e^{4t}$$

$\forall t\in R$ $e^t \neq 0$, we reduce the above equation to

$$g'(t)-2g(t)=3e^{2t}$$

Multiply $u(t)=e^{-2t}$, we get

$$e^{-2t}g'(t)-2e^{-2t}g(t)=3.$$

$$(e^{-2t}g(t))'=3$$

$$g(t)=(3t+c)e^{2t}$$

where C is constant.

If $w(f,g)=t^2e^t$, are they linearly dependent?

Solution: $w(f,g)\neq 0$ except one point

$\forall I\in R$, I is an open interval

$t=0$. then f, g are linear independ on I

2 Are y_1 and y_2 the solutions of given equation? Do they constitute a fundamental set of solution?

1) $y''-2y'+y=0$ $y_1(t)=e^t$ $y_2(t)=t\,e^t$

Solution: $y_1(t)''=e^t$ $y_1(t)'=e^t$ $y_1''-2y_1'+y_1=e^t-2e^t+e^t=0$

On the other hand $y_2'(t)=e^t+te^t$ $y_2''=2e^t+te^t$

$y_2''-2y_2'+y_2=2e^t+te^t-2e^t-2te^t+te^t=0$

Therefore both $y_1(t)$ and $y_2(t)$ are solutions.

Now we examine the wronskian of y_1 and y_2

$$W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^t(e^t + te^t) - e^t \cdot te^t$$

$$= (e^t)^2 > 0$$

So we say $y_1(t)$ and $y_2(t)$ are linear indepent to be a fundamental set of solution

2). $x^2y'' - x(x+2)y' + (x+2)y = 0 \quad x > 0, \; y_1(x) = x \; \cdot \; y_2(x) = xe^x$

Solution: $y_1'(x) = 1 \quad y_1''(x) = 0 \quad \Rightarrow \; -x(x+2) + (x+2)x = 0$

$y_2'(x) = xe^x + e^x \quad y_2'' = xe^x + 2e^x \Rightarrow$

$$x^2(xe^x + 2e^x) - x(x+2)(xe^x + e^x) + (x+2)xe^x$$

$$= x^2(xe^x + 2e^x) - x^2(xe^x + e^x) - 2x(xe^x + e^x) + (x+2)xe^x$$

$$= x^2e^x - 2x^2e^x - 2xe^x + x^2e^x + 2xe^x = 0$$

Therefore both $y_1(x)$ and $y_2(x)$ are solutions

The wronskian

$$W = \begin{vmatrix} x & xe^x \\ 1 & xe^x + e^x \end{vmatrix} = x^2e^x \neq 0$$

$\Rightarrow$ y_1 and y_2 can constitute a fundamental set of solution.

3. Solve the initial value problem

$$y''-y'-2y=0 \quad y(0)=\alpha \quad y'(0)=2$$

Then find α sothat the colution approach zero as $t\to\infty$

Solution: C.E $r^2-r-2=0 \Rightarrow r_1=-1 \quad r_2=2$

$$y(t)=C_1e^{-t}+C_2e^{2t}$$

$$y'(t)=-C_1e^{-t}+2C_2e^{2t}$$

Considen the initial value

$$\begin{cases} C_1+C_2=\alpha \\ -C_1+2C_2=2 \end{cases} \Rightarrow \begin{cases} C_1=\dfrac{2\alpha-2}{3} \\ C_2=\dfrac{\alpha+2}{3} \end{cases}$$

$$\Rightarrow y(t)=\frac{2\alpha-2}{3}e^{-t}+\frac{\alpha+2}{3}e^{2t}$$

To satify the requisite that $y(t)\to 0$ as $t\to\infty$, the coefficient of C_2 shoeld be zero. that is $\alpha=-2$

Deduced to $y(t)=\dfrac{2\alpha-2}{3}e^{-t}$

$$\lim_{t\to\infty} y(t)=\frac{2\alpha-2}{3}e^{-t}=0.$$

4. Find the solution of the given problem, describe it behavior as t increase

1). $y''+4y'+5y=0 \qquad y(0)=1 \quad y'(0)=0$

Solution: $r^2+4r+5=0 \Rightarrow r_1=-2+i \quad r_2=-2-i$

General colution: $y(t)=C_1e^{-2t}\cos t+C_2e^{-2t}\sin t$

$y(0)=1\Rightarrow C_1=1 \qquad y'(t)=-2e^{-2t}\cos t-e^{-2t}\sin t-2C_2e^{-2t}\sin t$

$y'(0) = -2e^{-2t} + C_2 e^{-2t} = 0 \Rightarrow C_2 = 2$

$\Rightarrow$ The solution $y(t) = e^{-2t}\cos t + 2e^{-2t}\sin t$

As $t \to \infty$ the solution $y(t)$ is decaying oscillation.

2). $y'' + y' + \frac{5}{4}y = 0 \qquad y(0) = 3. \quad y'(0) = 1$

C.E : $r^2 + r + \frac{5}{4} = 0 \Rightarrow r_1 = -\frac{1}{2} + i, \quad r_2 = -\frac{1}{2} - i$

$y = C_1 e^{-\frac{t}{2}}\cos t + C_2 e^{-\frac{t}{2}}\sin t$

consider the initial condition $y(0) = 3 \quad y'(0) = 1$

$y(0) = 3 \Rightarrow C_1 = 3; \quad y'(0) = 1 \Rightarrow C_2 = \frac{5}{2}$

$\Rightarrow \quad y = 3e^{-\frac{t}{2}}\cos t + \frac{5}{2}e^{-\frac{t}{2}}\sin t$

As $t \to \infty$, $y(t)$ is decaying oscillation

3). $y'' - 2y' + 5y = 0 \qquad y(\frac{\pi}{2}) = 0, \quad y'(\frac{\pi}{2}) = 2$

C.E: $r^2 - 2y + 5 = 0 \Rightarrow r_1 = 1 + 2i \quad r_2 = 1 - 2i$

$y = C_1 e^t \cos 2t + C_2 e^t \sin 2t$

$y(\frac{\pi}{2}) = 0 \Rightarrow C_1 = 0$

$y'(\frac{\pi}{2}) = 0 \Rightarrow C_2 = -e^{-\frac{\pi}{2}}$

$y = -e^{t-\frac{\pi}{2}}\sin 2t$

as $t \to \infty$, y is growing oscillation.

5\. 1) $y'' + 4y = t^2 + 3e^t$

C.E : $r^2 + 4 = 0 \Rightarrow r_1 = -2i \quad r_2 = 2i$

$$y = C_1 \cos 2t + C_2 \sin 2t$$

Let $Y(t) = At^2 + Bt + C + De^t$

$Y'(t) = 2At + B + De^t$

$Y''(t) = 2A + De^t$

$\Rightarrow (2A + De^t) + 4(At^2 + Bt + C + De^t) = t^2 + 3e^t$

$$\Rightarrow \begin{cases} B = 0 \\ 2A + 4C = 0 \\ 4A = 1 \\ 5D = 3 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{4} \\ B = 0 \\ C = -\frac{1}{8} \\ D = \frac{3}{5} \end{cases}$$

$\Rightarrow \quad y = C_1 \cos 2t + C_2 \sin 2t + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t$

2) $y'' - 2y' + y = te^t + 4$

C.E : $r^2 - 2r + 1 = 0 \Rightarrow r_1 = r_2 = 1$

$$y = C_1 e^t + C_2 t e^t$$

Let $Y(t) = t^2(At + B)e^t + C$

$Y'(t) = A(3t^2 e^t + t^3 e^t) + B(2t + t^2)e^t$

$Y''(t) = A(6t + 6t^2 + t^3)e^t + B(2 + 4t + t^2)e^t$

$\Rightarrow 6Ate^t + C = te^t + 4$

$\Rightarrow A = \frac{1}{6} \quad B = 0 \quad C = 4$

So $Y(t) = C_1 e^t + C_2 t e^t + \frac{1}{6}t^3 e^t + 4$

3. $y'' + 2y' + y = 2e^{-t}$

$C.E: \ r^2 + 2r + 1 = 0$

$\Rightarrow r_1 = r_2 = -1$

$y(t) = C_1 e^{-t} + C_2 t e^{-t}$

$Y(t) = At^2 e^{-t}$

$Y'(t) = A(2t - t^2)e^{-t}$

$Y''(t) = A(t^2 - 4t + 2)e^{-t}$

$\Rightarrow A = 1 \quad \Rightarrow y(t) = C_1 e^{-t} + C_2 t e^{t} + t^2 e^{-t}$

4). $y'' + 4y = 3\sin 2t \qquad y(0) = 2 \quad y'(0) = 1$

$C.E: \ r^2 + 4 = 0 \qquad r_1 = -2 \quad r_2 = 2$

$y(t) = C_1 \cos 2t + C_2 \sin 2t$

Let $Y(t) = t(A\cos 2t + B\sin 2t)$

$Y''(t) = -4A\sin 2t - 4At\cos 2t + 4B\cos 2t - 4Bt\sin 2t$

$\Rightarrow A = -\frac{3}{4} \qquad B = 0 \quad y(t) = C_1\cos 2t + C_2 \sin 2t - \frac{3}{4} t\cos 2t$

as $y(0) = 2$, $y'(0) = 1 \Rightarrow \begin{cases} C_1 = 2 \\ C_2 = \frac{7}{8} \end{cases}$

$\Rightarrow y(t) = 2\cos 2t + \frac{7}{8}\sin 2t - \frac{3}{4} t\cos 2t$

5

1) $y''+3y'+2y = e^{3t}(t^2+1)\cos 2t + 4e^{-2t}\sin t + 5$

$C.E: r^2+3r+2=0 \Rightarrow r_1=-1 \quad r_2=-2$

$$Y(t) = (At^2+Bt+C)e^{3t}\cos 2t + (Dt^2+Et+F)e^{3t}\sin 2t$$
$$+ Ge^{-2t}\cos t + He^{2t}\sin t + I.$$

2) $y''+3y' = 2t+5+te^{-3t}+\cos 4t$

$CE: r^2+3=0 \Rightarrow r_1=0, r_2=-3$

$y = c_1 + c_2 e^{-3t}$

$$Y(t) = t(At+B) + t(Ct+D)e^{-3t} + \cos 4t + \sin 4t.$$