#11 (a) $f(x) = (x-1)(\sin x - \cos x) = 0$, then we obtain the fixed points are $x^* = 1$, $x_k^* = \frac{\pi}{4} + k\pi$, $k = 0, \pm 1, \cdots$

Here, $f'(x) = (\sin x - \cos x) + (x-1)(\cos x + \sin x)$

$$= (x-2)\cos x + x \sin x$$

① $f'(x^*) = f'(1) = \sin 1 - \cos 1 > 0$, so $x^* = 1$ is linearly unstable.

② $f'(x_k^*) = (-1)^k \sqrt{2}\left(\frac{\pi - 4}{4} + k\pi\right)$

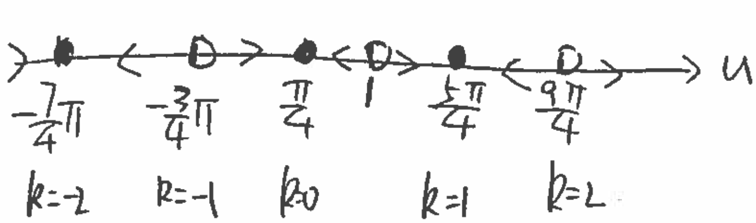
When $k < 0$, we have $f'(x_k^*) > 0$, $k = 2n+1$, $n = -1, -2, \cdots$

and $f'(x_k^*) < 0$, $k = 2n$, $n = -1, -2, \cdots$

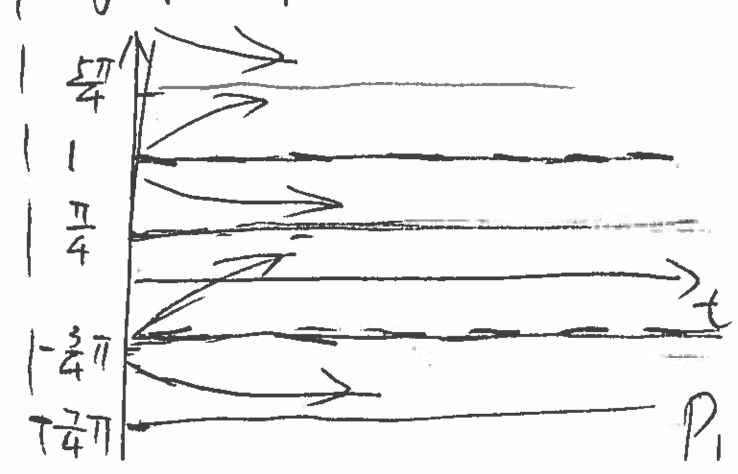
When $k \geq 0$, we have $f'(\frac{\pi}{4}) = \sqrt{2} \cdot \left(\frac{\pi - 4}{4}\right) < 0$,

$f'(x_k^*) > 0$, $k = 2n$, $n = 1, 2, \cdots$ and $f'(x_k^*) < 0$, $k = 2n-1$, $n = 1, 2, \cdots$

Thus, the phase portrait is



the graph of $x(t) - t$ is



(b). $f(x) = (x^2 - 7x + 6)(6 - x) = 0 \Rightarrow$

$x_1^* = 1$ and $x_2^* = 6$

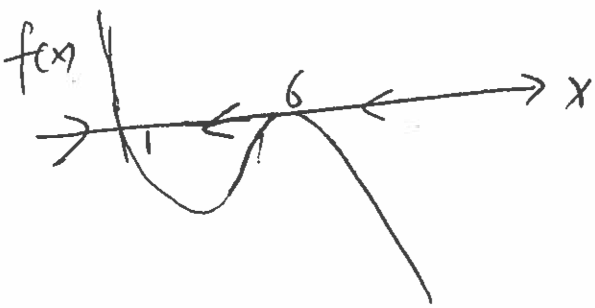
Here, $f'(x) = (2x-7)(6-x) - (x^2 - 7x + 6) = -3x^2 + 26x - 48$

① $f'(x_1^*) = f'(1) = -3 + 26 - 48 = -25 < 0$

So $x_1^* = 1$ is linearly stable.

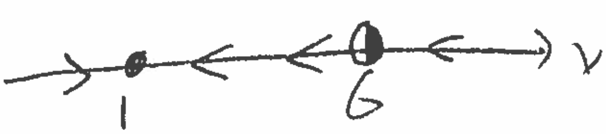
② $f'(x_2^*) = f'(6) = 0$, linear stability analysis fails.

But using a graphical argument, we obtain

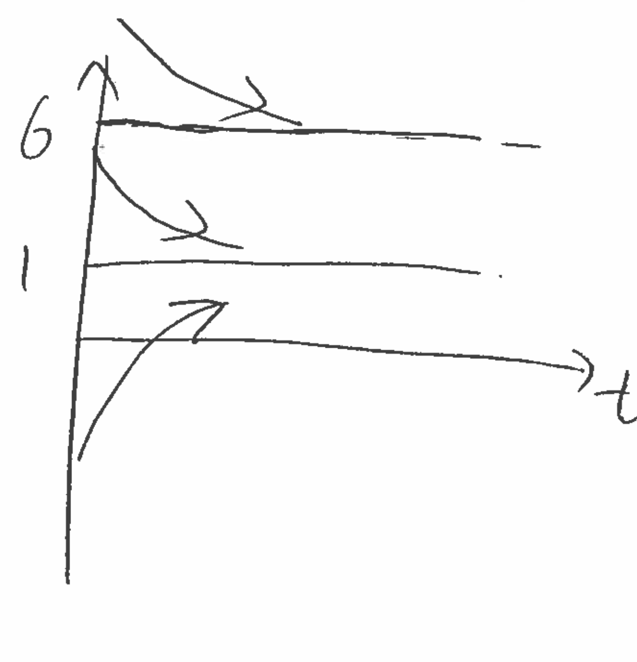


So $x_2^* = 6$ is half stable and half unstable.

The phase portrait is



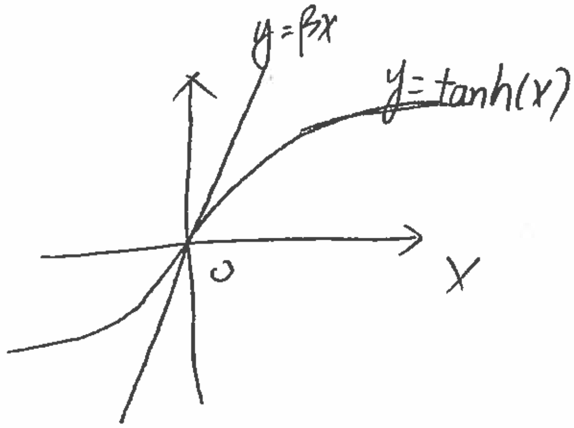
The graph of $x(t) - t$ is



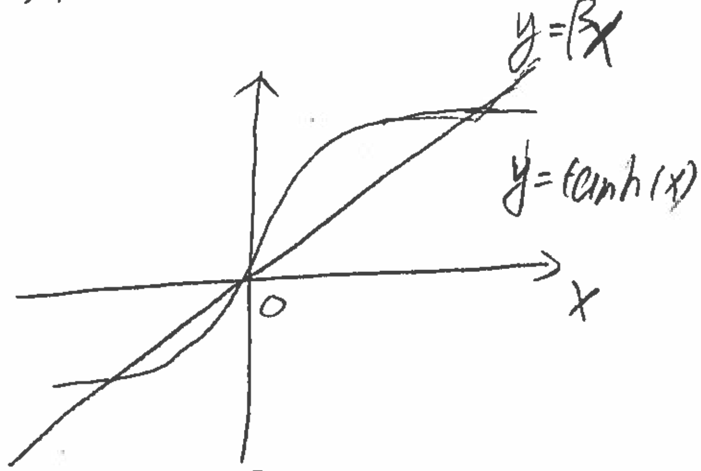
(C) $f(x) = \tanh(x) - \beta x = 0$, i.e., $\tanh(x) = \beta x$ $\beta > 0$.

The fixed points are intersections of the line $y = \beta x$, $\beta > 0$

and the curve $y = \tanh(x)$.



$\beta > 1$



$1 > \beta > 0$.

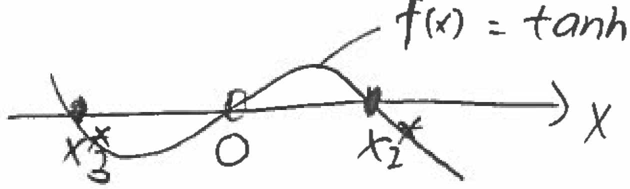
Thus, $\beta > 1$, only one fixed point $x^* = 0$.

$0 < \beta < 1$, three fixed points, $x_1^* = 0$, $x_2^* > 0$, $x_3^* < 0$

By linear stability analysis, we have $f'(x) = \text{sech}^2(x) - \beta$.

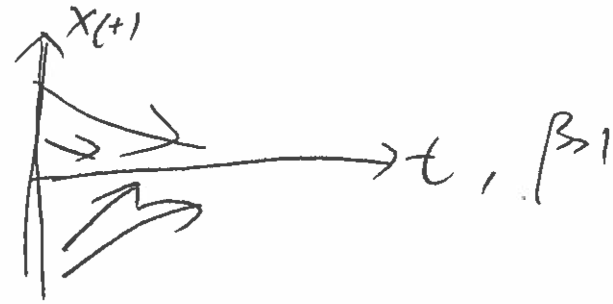
① $f'(0) = 1 - \beta$. so, $x_1^* = 0$ is linearly unstable when $0 < \beta < 1$, but stable when $\beta > 1$.

② With the help of graphical argument, we say $0 < \beta < 1$ x_2^* & x_3^* are stable but $x_1^* = 0$ is unstable



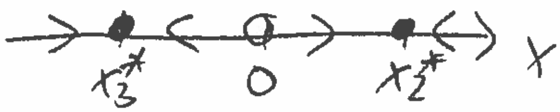
Thus the phase portrait is

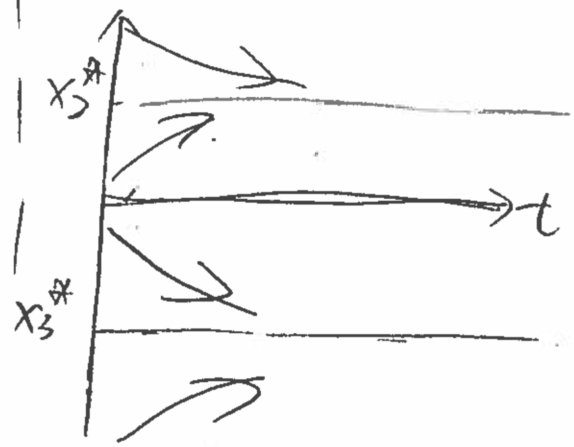
$\beta > 1$



, $\beta > 1$

$0 < \beta < 1$





, $0 < \beta < 1$

d> $f(x) = \mu x + x^3 - x^5 = 0$, $\mu > 0$, $\Rightarrow$

$x_1^* = 0$, $x_2^* = \sqrt{\frac{1+\sqrt{1+4\mu}}{2}}$, $x_3^* = -\sqrt{\frac{1+\sqrt{1+4\mu}}{2}}$

$f'(x) = \mu + 3x^2 - 5x^4$. so, for the stability, we have

① $f'(x_1^*) = f'(0) = \mu > 0$, so $x_1^* = 0$ is linearly unstable

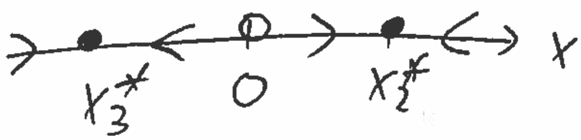
② $f'(x_2^*) = \mu + 3(x_2^*)^2 - 5(x_2^*)^4 = -2(x_2^*)^2 - 4\mu < 0$

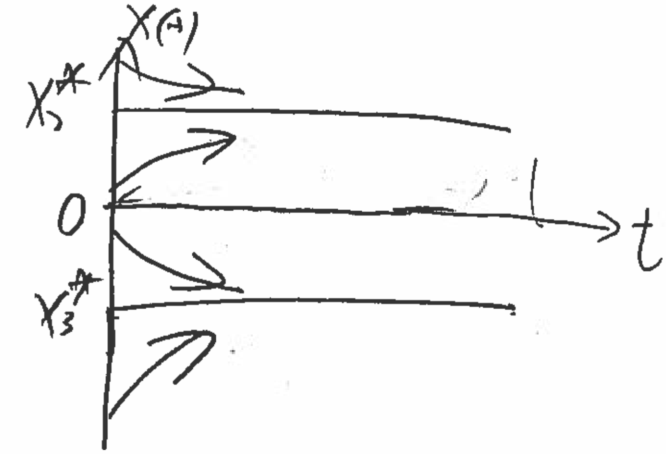
③ $f'(x_3^*) = -2(x_3^*)^2 - 4\mu < 0$

so, x_2^*, x_3^* are stable

D.

Thus, the phase portrait is





#2.(a) $f(x) = x(1-rx+x^2) \Rightarrow$

If $r^2<4$, i.e., $-2<r<2$, there is only one fixed point $x_1^*=0$

If $r^2=4$, i.e., $r=\pm 2$, there are two fixed points:

$x_1^*=0$, $x_2^*=1$, when $r=2$, OR $x_1^*=0$, $x_2^*=-1$, when $r=-2$.

If $r^2>4$, i.e., $r>2$ or $r<-2$, there are three fixed points

$$x_1^*=0,\quad x_2^*=\frac{r+\sqrt{r^2-4}}{2} \text{ and } x_3^*=\frac{r-\sqrt{r^2-4}}{2}.$$

As for the stability, we have $f'(x)=1-2rx+3x^2$

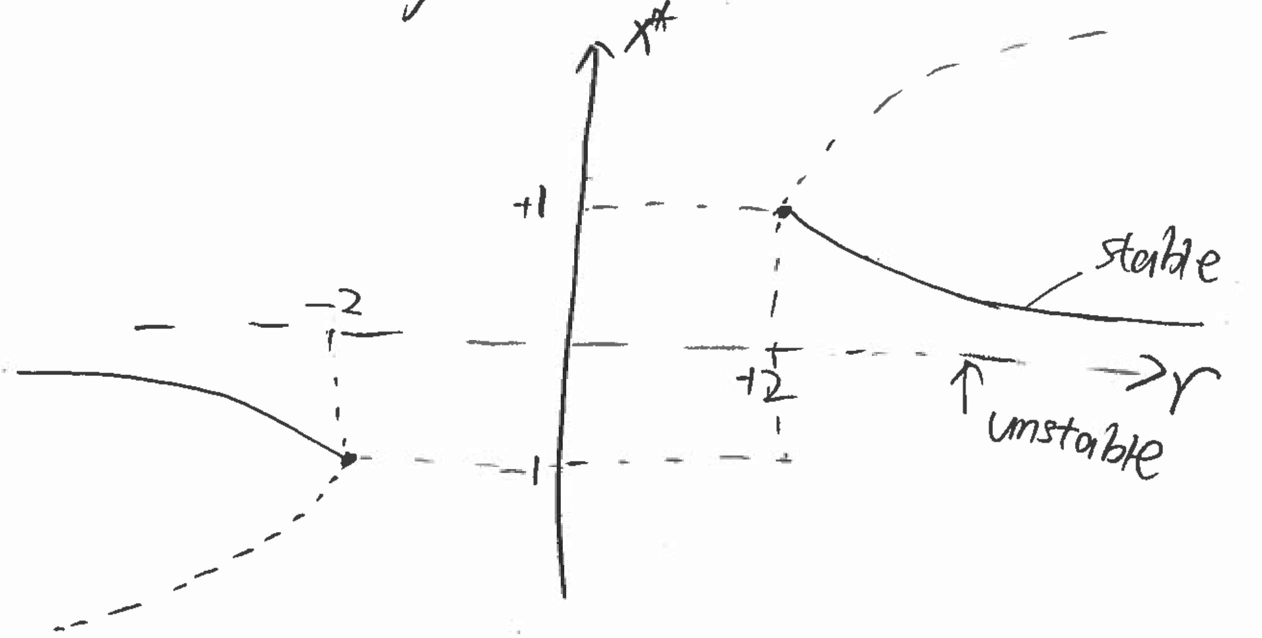
$$f'(x_1^*)=1; \quad f'(x_2^*)=\frac{r+\sqrt{r^2-4}}{2}\cdot\sqrt{r^2-4} = \begin{cases} >0 & (r>2) \\ <0 & (r<-2) \end{cases}$$

$$f'(x_3^*)=\frac{r-\sqrt{r^2-4}}{2}\cdot\left(-\sqrt{r^2-4}\right) = \begin{cases} <0 & (r>2) \\ >0 & (r<-2) \end{cases}$$

D

Thus, a saddle-node bifurcation occurs at $r_c = \pm 2$

and the diagram is:



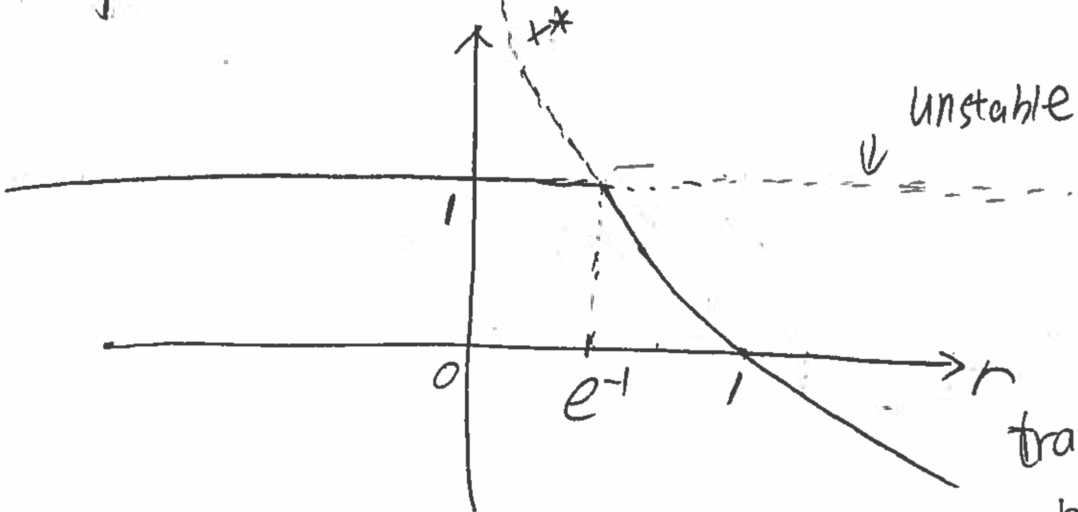
(b) $f(x) = (x-1)(r - e^{-x}) = 0 \Rightarrow$

- If $r \le 0$, there is only one fixed point $x^* = 1$.
- If $0 < r < e^{-1}$, there are two: $x_1^* = 1$, $x_2^* = -\ln r$
- If $r = e^{-1}$, there is only one: $x_1^* = 1$
- If $r > e^{-1}$, there are two: $x_1^* = 1$, $x_2^* = -\ln r$

As for the stability, we have $f'(x) = r - 2e^{-x} + xe^{-x}$

$$f'(1) = r - e^{-1} = \begin{cases} >0, & r > e^{-1} \\ =0, & r = e^{-1} \\ <0, & r < e^{-1} \end{cases}, \quad f'(-\ln r) = r \cdot \left(\ln \frac{1}{re}\right) = \begin{cases} >0, & 0 < r < e^{-1} \\ =0, & r = e^{-1} \\ <0, & r > e^{-1} \end{cases}$$

Thus, a bifurcation occurs at $r_c = e^{-1}$, and the diagram is



transcritical bifurcation

(c) $f(x) = rx - \frac{x}{1+x^2} = 0 \Rightarrow$

If $r \le 0$, there is only one fixed point: $x^* = 0$.

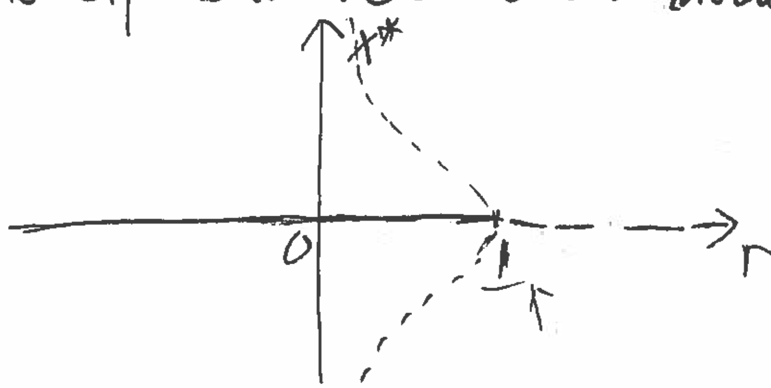
If $0 < r < 1$, there are three: $x_1^* = 0$, $x_2^* = \sqrt{\frac{1-r}{r}}$, $x_3^* = -\sqrt{\frac{1-r}{r}}$.

If $1 \le r$, there is only one: $x_1^* = 0$.

As for ~~the~~ stability, we have $f'(x) = r - \frac{1}{1+x^2} + \frac{2x^2}{(1+x^2)^2}$.

$f'(0) = r - 1$, $f'(x_2^*) = 2r(1-r) > 0$, $f'(x_3^*) = 2r(1-r) > 0$

Thus, the bifurcation occurs at $r_2 = 1$, the diagram is



$r_2 = 1$ --- subcritical pitchfork bifurcation

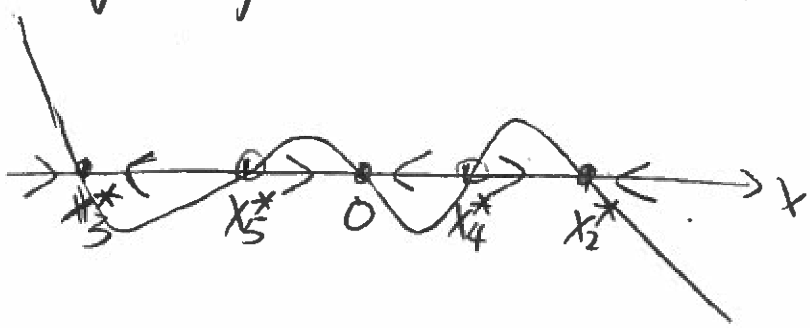
(d) : $f(x) = rx + x^3 - x^5 = 0$. As we did in 1(d), when $r \geq 0$, there are three fixed points: $x_1^* = 0$ unstable $x_2^* = \sqrt{\frac{1+\sqrt{1+4r}}{2}}$ stable and $x_3^* = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$ stable.

Now, let's consider the case where $r < 0$.

If $-\frac{1}{4} < r < 0$, there are five fixed points:

$x_1^* = 0$, $x_2^* = \sqrt{\frac{1+\sqrt{1+4r}}{2}}$, $x_3^* = -\sqrt{\frac{1+\sqrt{1+4r}}{2}}$, $x_4^* = \sqrt{\frac{1-\sqrt{1+4r}}{2}}$, $x_5^* = -\sqrt{\frac{1-\sqrt{1+4r}}{2}}$

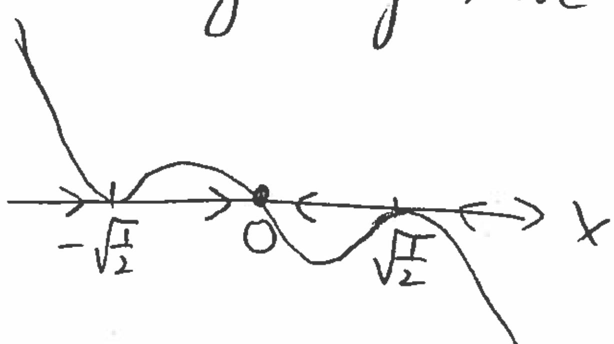
By linear stability analysis, we obtain the phase portrait as



If $r = -\frac{1}{4}$, there are three fixed points:

$x_1^* = 0$, $x_2^* = \sqrt{\frac{1}{2}}$, $x_3^* = -\sqrt{\frac{1}{2}}$

By linear stability analysis, we obtain the phase portrait as



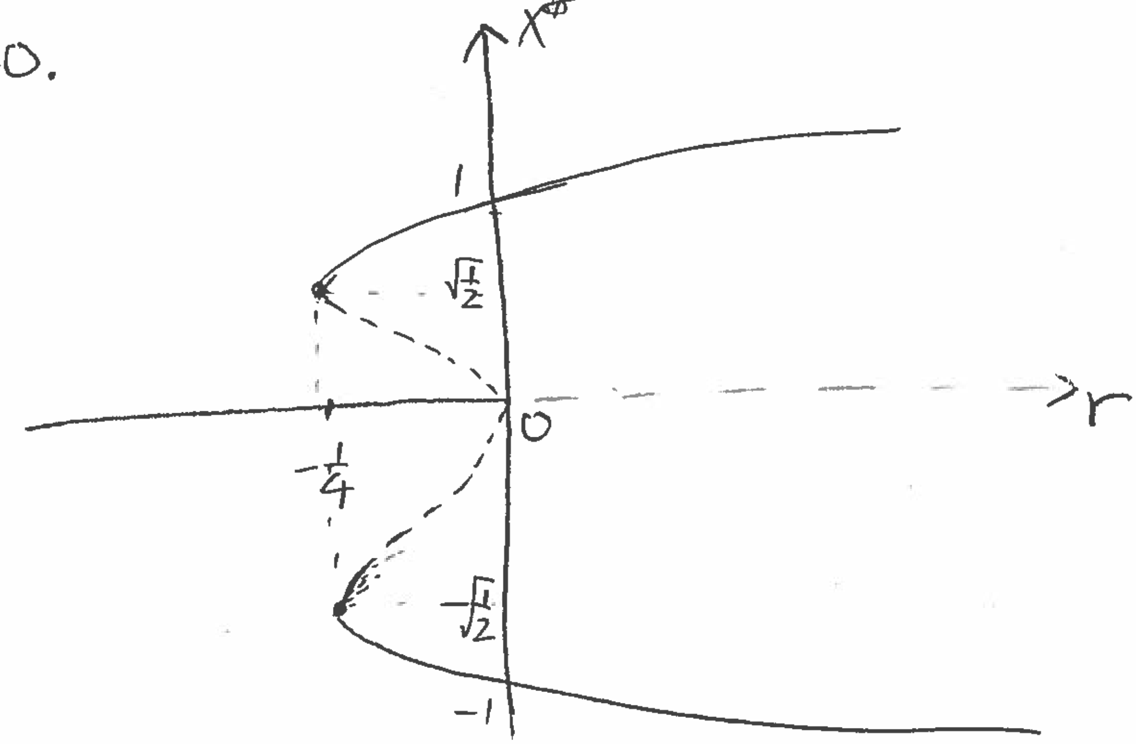
If $r < -\frac{1}{4}$, there is only one fixed point $x^* = 0$, which is stable

Pc

Thus, a saddle-node bifurcation occurs at

$r_1 = -\frac{1}{4}$, and a subcritical pitchfork bifurcation occurs at

$r_2 = 0$.



P2