MEMORIAL UNIVERSITY

DEPARTMENT OF MATHEMATICS & STATISTICS

Solution

TEST 2 | Math 3100 | WINTER 2018

Last Name: | **First name:** | **Student ID:**

1. For each of the following linear systems, Sketch the phase portrait.

[12] (a) $\begin{cases} x' = 4x - y \\ y' = 2x + y \end{cases}$ (b) $\begin{cases} x' = 3x - 4y \\ y' = x - y \end{cases}$.

Sol: (a). $A = \begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$ $|A-\lambda I| = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 4 + 2$

$= \lambda^2 - 5\lambda + 6 = 0.$

$= (\lambda-2)(\lambda-3) = 0$

$\lambda_1 = 2.\ \lambda_2 = 3.$

when $\lambda_1 = 2$, $(A-\lambda_1 I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

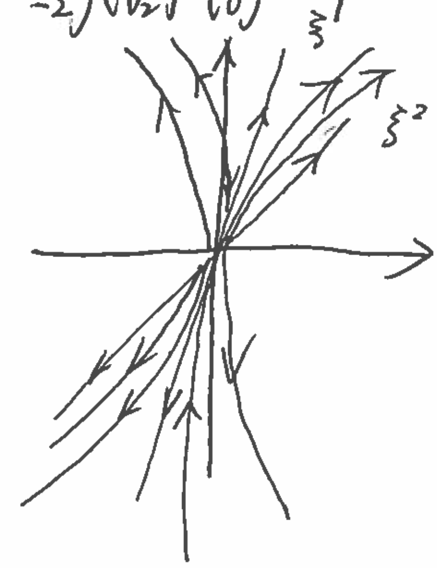
$\xi^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

when $\lambda_2 = 3$. $(A-\lambda_2 I)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \xi^2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$

Since $\lambda_1 > 0$, $\lambda_2 > 0$, we have the phase portrait.

(0, 0) is a unstable node.



(b). (0,0) is the fixed point.

$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$. $|A-\lambda I| = \begin{vmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 + 4 = \lambda^2 - 2\lambda + 1$

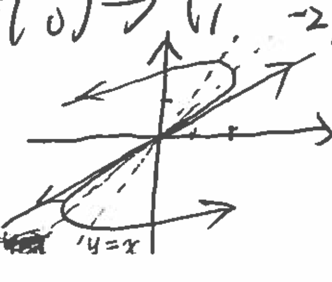
$= 0.$

$\lambda_1 = \lambda_2 = 1$. repeated.

This is a degenarated case.

$\cancel{|A-\lambda I|} = \begin{pmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$

$\xi^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. phase portrait:



when $x = y \Rightarrow y' = 0.$

when $y = \frac{3}{4}x \Rightarrow x' = 0$

Solution to Test 2.

2. For each of the following system, find the fixed points, classify them. Sketch the phase portrait.

[24] (a) $\begin{cases} x' = y + x - x^3 \\ y' = -y \end{cases}$ (b) $\begin{cases} x' = xy - 1 \\ y' = x - y^3 \end{cases}$ (c) $\begin{cases} x' = x(3 - 2x - y) \\ y' = y(2 - x - y) \end{cases}$.

(a). $\begin{cases} y + x - x^3 = 0 \\ -y = 0 \end{cases} \Rightarrow (0,0),\ (1,0),\ (-1,0)$

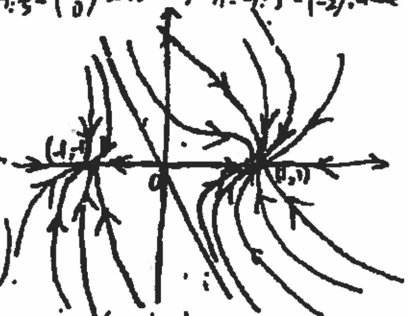
Jacobian matrix $A = \begin{pmatrix} 1-3x^2 & 1 \\ 0 & -1 \end{pmatrix}$

$A|_{(0,0)} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$: $\lambda = \pm 1$ saddle. $\lambda = 1$: $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ unstable; $\lambda = -1$: $\xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, stable

$A|_{(1,0)} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$: $\lambda = -1, -2$ stable node

$A|_{(-1,0)} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$: $\lambda = -1, -2$. stable node.

$\lambda = -1$: $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\lambda = -2$: $\xi^{(2)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

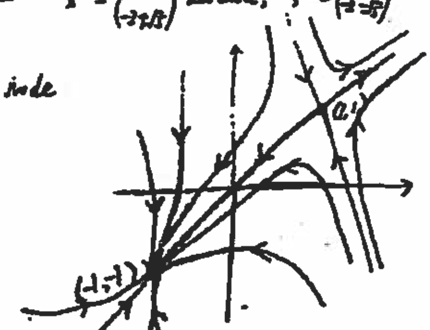


(b) $\begin{cases} xy - 1 = 0 \\ x - y^3 = 0 \end{cases} \Rightarrow (1,1),\ (-1,-1)$ Jacobian matrix $A = \begin{pmatrix} y & x \\ 1 & -3y^2 \end{pmatrix}$

$A|_{(1,1)} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$. $\lambda = -1 \pm \sqrt{5}$ saddle $\xi^{(1)} = \begin{pmatrix} 1 \\ -2+\sqrt{5} \end{pmatrix}$ unstable, $\xi^{(2)} = \begin{pmatrix} 1 \\ -2-\sqrt{5} \end{pmatrix}$ stable

$A|_{(-1,-1)} = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix}$: $\lambda = -2$ stable node

$\lambda = -2$: $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



(c). $\begin{cases} x(3-2x-y) = 0 \\ y(2-x-y) = 0 \end{cases}$ $(0,0)$ $(1,1)$. $(0,2)$ $(\frac{3}{2},0)$.

$A = \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}$.

$A|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$: $\lambda = 2, 3$ unstable node

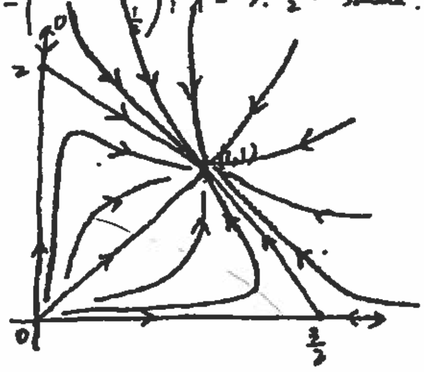
$A|_{(1,1)} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$: $\lambda = \frac{-3 \pm \sqrt{5}}{2} < 0$ stable node

$A\big|_{(0,2)} = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$: $\lambda = 1, -2$. saddle. $\xi^{(1)} = \begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}$ unstable

$\xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ stable.

$A\big|_{(\frac{3}{2},0)} = \begin{pmatrix} -3 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$; $\lambda = -3, \frac{1}{2}$. saddle. $\xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ stable

$\xi^{(2)} = \begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}$ unstable.



[8] 3. Consider the system $\begin{cases} x' = -y - x^3 \\ y' = x. \end{cases}$ Show that the origin is a spiral, although the linearization predicts a center.

- Linear system: $A = \begin{pmatrix} -2x^2 & -1 \\ 1 & 0 \end{pmatrix}$, $A|_{(0,0)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

  $|A - \lambda I| = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$, center.

- non-linear system by polar coordinates:

$$\begin{cases} r' = \dfrac{xx' + yy'}{r} = \dfrac{-x^4}{r} = -r^3\cos^4\theta \\ \theta' = \dfrac{y'x - yx'}{r^2} = 1 + r^2\cos^3\theta\sin\theta \end{cases}$$

since $r > 0$, $r' = -r^3\cos^4\theta \le 0$, $r' \neq 0$ when $\cos\theta \neq 0$,

the origin is a stable spiral.

[6] 4. Transfer the system $\begin{cases} x' = \mu x - y - xy^2 \\ y' = x + \mu y + y^3 \end{cases}$ into a polar system. If $\mu > 0$, what is the limit of every orbit as $t \to \infty$?

Sol: Recall $r' = \dfrac{xx' + yy'}{r}$, $\theta' = \dfrac{y'x - yx'}{r^2}$

This gives
$$\begin{cases} r' = \mu r + \frac{1}{r} y^2(y^2 - x^2) = \mu r + r^3\sin^2\theta(\sin^2\theta - \cos^2\theta) \\ \theta' = \frac{1}{r^2}(x^2 + y^2 + 2xy^3) = 1 + 2r^2\cos\theta\sin^3\theta. \end{cases}$$

If $\mu > 0$, we have.

$$\begin{aligned} r' &= r(\mu + r^2\sin^2\theta(\sin^2\theta - \cos^2\theta)) \\ &= r(\mu - r^2\sin^2\theta(\cos^2\theta - \sin^2\theta)) \\ &= r\left(\sqrt{\mu} - r\sqrt{\sin^2\theta(\cos^2\theta - \sin^2\theta)}\right) \cdot \left(\sqrt{\mu} + r\sqrt{\sin^2\theta(\cos^2\theta - \sin^2\theta)}\right) \end{aligned}$$

This implies, the orbit will tend to a curve defined by $\sqrt{\mu} - r\sqrt{\sin^2\theta(\cos^2\theta - \sin^2\theta)} = 0$, as $t \to \infty$