EXERCISES FOR CHAPTER 10

Note: Many of these exercises ask you to use a computer. Feel free to write your own programs, or to use commercially available software. The programs in *MacMath* (Hubbard and West 1992) are particularly easy to use.

10.1 Fixed Points and Cobwebs

(Calculator experiments) Use a pocket calculator to explore the following maps. Start with some number and then keep pressing the appropriate function key; what happens? Then try a different number—is the eventual pattern the same? If possible, explain your results mathematically, using a cobweb or some other argument.

√ **10.1.1** $x_{n+1} = \sqrt{x_n}$ **10.1.2** $x_{n+1} = x_n^3$

10.1.3 $x_{n+1} = \exp x_n$ **10.1.4** $x_{n+1} = \ln x_n$

10.1.5 $x_{n+1} = \cot x_n$ √ **10.1.6** $x_{n+1} = \tan x_n$

10.1.7 $x_{n+1} = \sinh x_n$ √ **10.1.8** $x_{n+1} = \tanh x_n$

10.1.9 Analyze the map $x_{n+1} = 2x_n/(1+x_n)$ for both positive and negative x_n.

10.1.10 Show that the map $x_{n+1} = 1 + \frac{1}{2}\sin x_n$ has a unique fixed point. Is it stable?

√ **10.1.11** (Cubic map) Consider the map $x_{n+1} = 3x_n - x_n^3$.

a) Find all the fixed points and classify their stability.
b) Draw a cobweb starting at $x_0 = 1.9$.
c) Draw a cobweb starting at $x_0 = 2.1$.
d) Try to explain the dramatic difference between the orbits found in parts (b) and (c). For instance, can you prove that the orbit in (b) will remain bounded for all n? Or that $|x_n| \to \infty$ in (c)?

√ **10.1.12** (Newton's method) Suppose you want to find the roots of an equation $g(x) = 0$. Then ***Newton's method*** says you should consider the map $x_{n+1} = f(x_n)$, where

$$f(x_n) = x_n - \frac{g(x_n)}{g'(x_n)}.$$

a) To calibrate the method, write down the "Newton map" $x_{n+1} = f(x_n)$ for the equation $g(x) = x^2 - 4 = 0$.
b) Show that the Newton map has fixed points at $x^* = \pm 2$.
c) Show that these fixed points are *superstable*.
d) Iterate the map numerically, starting from $x_0 = 1$. Notice the extremely rapid convergence to the right answer!

10.1.13 (Newton's method and superstability) Generalize Exercise 10.1.12 as fol-

10.3.3 Analyze the long-term behavior of the map $x_{n+1} = rx_n/(1+x_n^2)$, where $r > 0$. Find and classify all fixed points as a function of r. Can there be periodic solutions? Chaos?

10.3.4 (Quadratic map) Consider the ***quadratic map*** $x_{n+1} = x_n^2 + c$.

a) Find and classify all the fixed points as a function of c.
b) Find the values of c at which the fixed points bifurcate, and classify those bifurcations.
c) For which values of c is there a stable 2-cycle? When is it superstable?
d) Plot a partial bifurcation diagram for the map. Indicate the fixed points, the 2-cycles, and their stability.

10.3.5 (Conjugacy) Show that the logistic map $x_{n+1} = rx_n(1-x_n)$ can be transformed into the quadratic map $y_{n+1} = y_n^2 + c$ by a linear change of variables, $x_n = ay_n + b$, where a, b are to be determined.

(One says that the logistic and quadratic maps are "conjugate." More generally, a ***conjugacy*** is a change of variables that transforms one map into another. If two maps are conjugate, they are equivalent as far as their dynamics are concerned; you just have to translate from one set of variables to the other. Strictly speaking, the transformation should be a homeomorphism, so that all topological features are preserved.)

10.3.6 (Cubic map) Consider the cubic map $x_{n+1} = f(x_n)$, where $f(x_n) = rx_n - x_n^3$.

a) Find the fixed points. For which values of r do they exist? For which values are they stable?
b) To find the 2-cycles of the map, suppose that $f(p) = q$ and $f(q) = p$. Show that p, q are roots of the equation $x(x^2 - r + 1)(x^2 - r - 1)(x^4 - rx^2 + 1) = 0$ and use this to find all the 2-cycles.
c) Determine the stability of the 2-cycles as a function of r.
d) Plot a partial bifurcation diagram, based on the information obtained.

10.3.7 (A chaotic map that can be analyzed completely) Consider the ***decimal shift map*** on the unit interval given by

$$x_{n+1} = 10x_n \pmod 1.$$

As usual, "mod 1" means that we look only at the noninteger part of x. For example, $2.63 \pmod 1 = 0.63$.

a) Draw the graph of the map.
b) Find all the fixed points. (Hint: Write x_n in decimal form.)
c) Show that the map has periodic points of all periods, but that all of them are unstable. (For the first part, it suffices to give an explicit example of a period-p point, for each integer $p > 1$.)
d) Show that the map has infinitely many aperiodic orbits.