45

4' 10.1.1. $x_{n+1} = \sqrt{x_n}$

eg: $x_0 = 2 \Rightarrow x_1 = 1.414$

$x_2 = 1.189$

$x_3 = 1.091$

$\vdots$

$x_n \to 1.$

Let $f(x) = \sqrt{x}$, then we can obtain:

$x = \sqrt{x} \Rightarrow x^* = 0, 1$ are fixed points

Since $f'(x) = \frac{1}{2\sqrt{x}}$, we have:

$|f'(1)| = \frac{1}{2} < 1. \Rightarrow x^* = 1$ is stable.

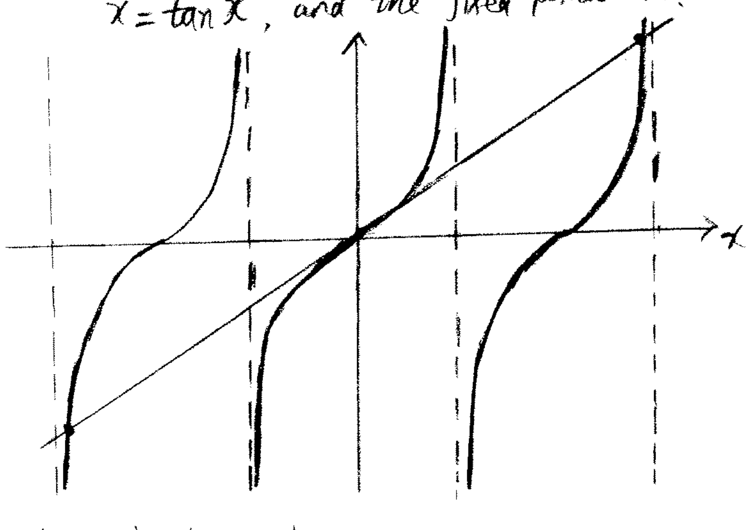
$x^* = 0$ is unstable.

$\therefore$ Any positive x_0 will converge to 1.

4' 10.1.6. $x_{n+1} = \tan x_n$

Let $f(x) = \tan x$, then

$x = \tan x$, and the fixed points are:



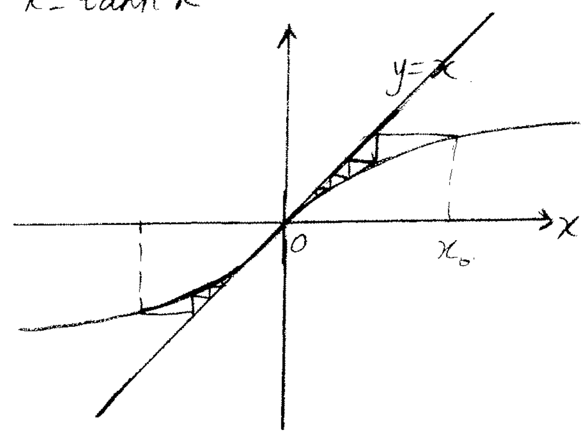
$|f'(x)| = |\sec^2 x| > 1 \quad (x \neq \frac{\pi}{2})$

so all the fixed points are unstable.

4' 10.1.8. $x_{n+1} = \tanh x_n$

Let $f(x) = \tanh x$, then

$x = \tanh x$



The fixed point is $x^* = 0$

$|f'(x^*)| = |1 - \tan^2 x^*| = |1 - 0| = 1$

$x^* = 0$ is stable.

9' 10.1.11

(a) Let $f(x) = 3x - x^3$, then

$x = 3x - x^3 \Rightarrow$

The fixed points are $x_1^* = 0$, $x_2^* = -\sqrt{2}$, $x_3^* = \sqrt{2}$.

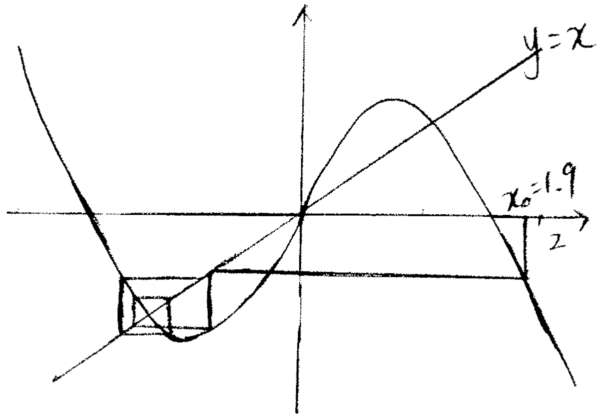
$f'(x) = 3 - 3x^2$

$|f'(x_1^*)| = 3 > 1, \quad |f'(x_2^*)| = |3 - 3\times 2| = 3 > 1$

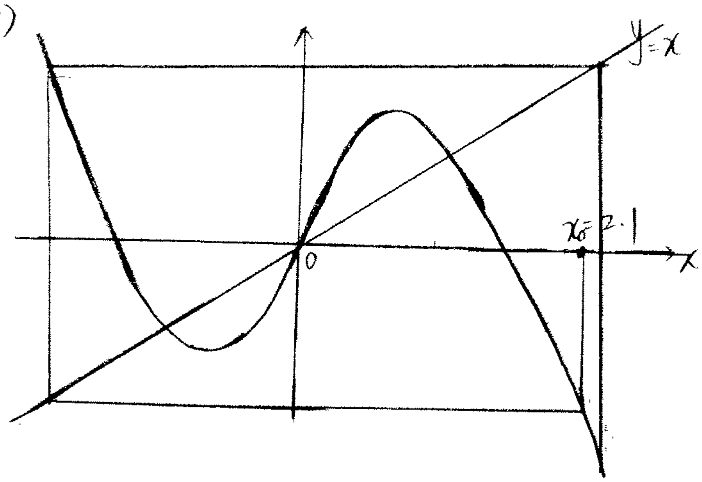
$|f'(x_3^*)| = 3 > 1$

$\therefore$ All the fixed points are unstable.

(b)



(c)



(d). $f'(x) = 3-3x^2=0 \Rightarrow x=\pm 1$

so $f(x)$ has maximum at $x=1$ and minimum at $x=-1$. $f(x)_{max} = f(1) = 2$, $f_{min} = f(-1) = -2$.

Since $f(2) = 6-8=-2$, and $f(-2) = -6+8=2$

we obtain that $f(x)$ maps $[-2,2] \to [-2,2]$.

Any $x_0 \in [-2,2]$ will remain in $[-2,2]$,

and if $|x_0| > 2$, then $|x_n| \to \infty$

10.1.12 6'

(a) $g(x) = x^2-4$, $g'(x) = 2x$

then $x_{n+1} = f(x_n) = x_n - \dfrac{x_n^2-4}{2x_n}$

(b). From $x = x - \dfrac{x^2-4}{2x}$, we obtain

$x^* = \pm 2$ are the fixed points.

(c) $f(x) = x - \dfrac{x^2-4}{2x}$,

$$f'(x) = 1 - \frac{4x^2-2x^2+8}{4x^2} = 1 - \frac{x^2+4}{2x^2}$$

So $\left|f'(\pm 2)\right| = \left|1 - \dfrac{4+4}{8}\right| = 0$

i.e. $x^* = \pm 2$ are superstable.

(d) $x_0 = 1$

$$x_1 = 1 - \frac{1-4}{2} = 2.5$$

$$x_2 = 1 - \frac{x_1^2-4}{2x_1} = 2.05$$

$$x_3 = 1 - \frac{x_2^2-4}{2x_2} = 2.00061 \quad \cdots$$

10.3.3 8'

$x_{n+1} = rx_n/(1+x_n^2)$

Let $f(x) = rx/(1+x^2)$, then

$$x = f(x) \Leftrightarrow \frac{rx}{1+x^2} = x$$

$$\Leftrightarrow x^3 + (1-r)x = 0 \Rightarrow x_1^* = 0,\ x_{2,3}^* = \pm\sqrt{r-1}\ (r>1)$$

i.e. For $0<r\leq 1$, one fixed point $x^*=0$

For $r>1$, three fixed points x_1^*, x_2^*, x_3^*

$$f'(x) = r\,\frac{1+x^2-2x^2}{(1+x^2)^2} = \frac{r(1-x^2)}{(1+x^2)^2}$$

$$|f'(0)| = \left|\frac{r(1-0)}{(1+0)^2}\right| = r$$

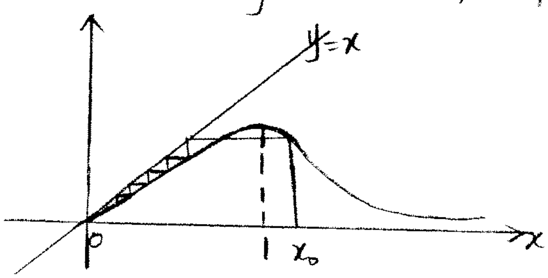
$\Rightarrow x_1^*=0$ is stable if $0<r<1$,

unstable if $r>1$

$$\left|f'(x_{2,3}^*)\right| = \left|\frac{r(1-r+1)}{(1+r-1)^2}\right| = \left|\frac{2r-r^2}{r^2}\right| = \left|\frac{2}{r}-1\right|$$

$\Rightarrow x_{2,3}^* = \pm\sqrt{r-1}$ are stable for $r>1$.

For $r=1$, we only have one fixed point $x^*=0$



So $x^*=0$ is stable for $r=1$

Therefore,

$x^*=0$ is stable if $0<r\le 1$

$x^*=0$ is unstable if $r>1$, and

$x_{2,3}^* = \pm\sqrt{r-1}$ is stable if $r>1$

So there are no periodic solutions or chaos for this system.

If $r\in(0,1]$, then any x_0 will converge to $x^*=0$; if $r\in(1,+\infty)$, then $x_{2,3}^*=\pm\sqrt{r-1}$ is stable.

10.3.6 10'

(a) $x_{n+1} = r x_n - x_n^3$

From $x = rx - x^3$, i.e. $x^3+(1-r)x=0$, we obtain that

$r\le 1$, one fixed point $x^*=0$;

$r>1$, three fixed points: $x_1^*=0$, $x_{2,3}^*=\pm\sqrt{r-1}$

$f(x)=rx-x^3$, $f'(x)=r-3x^2$

$|f'(0)| = |r| \Rightarrow x^*=0$ is $\begin{cases}\text{stable if } -1<r<1\\ \text{unstable if } |r|>1\end{cases}$

$|f'(x_{2,3}^*)| = |r-3(r-1)| = |3-2r|$

$\Rightarrow x_{2,3}^*$ are $\begin{cases}\text{stable if } 1<r<2\\ \text{unstable if } r>2\end{cases}$

(b) $p=f(q)$, $q=f(p)$

$\Rightarrow x = r(rx-x^3) - (rx-x^3)^3$

$\Rightarrow \underline{x[x^2-r+1][x^2-r-1][x^4-rx^2+1]=0}$

$\Rightarrow x_1^*=0$, $x_{2,3}^*=\pm\sqrt{r-1}$, $x_{4,5}^*=\pm\sqrt{r+1}$,

and $x^4-rx^2+1=0$

Let $t=x^2$, then $t^2-rt+1=0$

$$\Rightarrow t=\frac{r\pm\sqrt{r^2-4}}{2}$$

$x_{6,7,8,9}^* = \pm\left(\frac{r\pm\sqrt{r^2-4}}{2}\right)^{\frac{1}{2}}$ —— P.9.

(c). $\lambda = \left|\left(f(f(p))\right)'\right| = \left| f'(f(p)) \cdot f'(q) \right|$

$= \left| f'(q) \cdot f'(p) \right|$, $f'(x) = r - 3x^2$.

$\therefore \lambda = \left|(r-3q^2)(r-3p^2)\right|$

$= \left|\left(r - 3\frac{r+\sqrt{r^2-4}}{2}\right)\left(r - 3\frac{r-\sqrt{r^2-4}}{2}\right)\right|$

$= \left|9 - 2r^2\right|$

$\Rightarrow$ p, q are stable if $|9-2r^2| < 1$ i.e. $2 < r \leq \sqrt{5}$.

(d)