models should predict a *single* closed orbit, or perhaps finitely many, but not a continuous family of neutrally stable cycles. See the discussions in May (1972), Edelstein–Keshet (1988), or Murray (1989).

6.6 Reversible Systems

Show that each of the following systems is reversible, and sketch the phase portrait.

6.6.1 $\dot{x} = y(1-x^2)$, $\dot{y} = 1-y^2$

6.6.2 $\dot{x} = y$, $\dot{y} = x\cos y$

6.6.3 (Wallpaper) Consider the system $\dot{x} = \sin y$, $\dot{y} = \sin x$.

a) Show that the system is reversible.
b) Find and classify all the fixed points.
c) Show that the lines $y = \pm x$ are invariant (any trajectory that starts on them stays on them forever).
d) Sketch the phase portrait.

6.6.4 (Computer explorations) For each of the following reversible systems, try to sketch the phase portrait by hand. Then use a computer to check your sketch. If the computer reveals patterns you hadn't anticipated, try to explain them.

a) $\ddot{x} + (\dot{x})^2 + x = 3$ b) $\dot{x} = y - y^3$, $\dot{y} = x\cos y$ c) $\dot{x} = \sin y$, $\dot{y} = y^2 - x$

6.6.5 Consider equations of the form $\ddot{x} + f(\dot{x}) + g(x) = 0$, where f is an even function, and both f and g are smooth.

a) Show that the equation is invariant under the pure time-reversal symmetry $t \to -t$.
b) Show that the equilibrium points cannot be stable nodes or spirals.

6.6.6 (Manta ray) Use qualitative arguments to deduce the "manta ray" phase portrait of Example 6.6.1.

a) Plot the nullclines $\dot{x} = 0$ and $\dot{y} = 0$.
b) Find the sign of $\dot{x}$, $\dot{y}$ in different regions of the plane.
c) Calculate the eigenvalues and eigenvectors of the saddle points at $(-1, \pm 1)$.
d) Consider the unstable manifold of $(-1,-1)$. By making an argument about the signs of $\dot{x}$, $\dot{y}$, prove that this unstable manifold intersects the negative x-axis. Then use reversibility to prove the existence of a heteroclinic trajectory connecting $(-1,-1)$ to $(-1,1)$.
e) Using similar arguments, prove that another heteroclinic trajectory exists, and sketch several other trajectories to fill in the phase portrait.

6.6.7 (Oscillator with both positive and negative damping) Show that the system $\ddot{x} + x\dot{x} + x = 0$ is reversible and plot the phase portrait.

$$\omega = 1 + \varepsilon\phi' = 1 + \tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2). \tag{57}$$

Now for the physical interpretation. The Duffing equation describes the undamped motion of a unit mass attached to a nonlinear spring with restoring force $F(x) = -x - \varepsilon x^3$. We can use our intuition about ordinary linear springs if we write $F(x) = -kx$, where the spring stiffness is now dependent on x:

$$k = k(x) = 1 + \varepsilon x^2.$$

Suppose $\varepsilon > 0$. Then the spring gets *stiffer* as the displacement x increases—this is called a ***hardening spring***. On physical grounds we'd expect it to *increase* the frequency of the oscillations, consistent with (57). For $\varepsilon < 0$ we have a ***softening spring***, exemplified by the pendulum (Exercise 7.6.15).

It also makes sense that $r' = 0$. The Duffing equation is a conservative system and for all ε sufficiently small, it has a ***nonlinear center*** at the origin (Exercise 6.5.13). Since all orbits close to the origin are periodic, there can be no long-term change in amplitude, consistent with $r' = 0$. ■

Validity of Two-Timing

We conclude with a few comments about the validity of the two-timing method. The rule of thumb is that the one-term approximation x_0 will be within $O(\varepsilon)$ of the true solution x for all times up to and including $t \sim O(1/\varepsilon)$, assuming that both x and x_0 start from the same initial condition. If x is a periodic solution, the situation is even better: x_0 remains within $O(\varepsilon)$ of x for *all* t.

But for precise statements and rigorous results about these matters, and for discussions of the subtleties that can occur, you should consult more advanced treatments, such as Guckenheimer and Holmes (1983) or Grimshaw (1990). Those authors use the *method of averaging*, an alternative approach that yields the same results as two-timing. See Exercise 7.6.25 for an introduction to this powerful technique.

Also, we have been very loose about the sense in which our formulas approximate the true solutions. The relevant notion is that of *asymptotic* approximation. For introductions to asymptotics, see Lin and Segel (1988) or Bender and Orszag (1978).

EXERCISES FOR CHAPTER 7

7.1 Examples

Sketch the phase portrait for each of the following systems. (As usual, r, θ denote polar coordinates.)

7.1.1 $\dot{r}=r^3-4r$, $\dot{\theta}=1$ 7.1.2 $\dot{r}=r(1-r^2)(9-r^2)$, $\dot{\theta}=1$

7.1.3 $\dot{r}=r(1-r^2)(4-r^2)$, $\dot{\theta}=2-r^2$ 7.1.4 $\dot{r}=r\sin r$, $\dot{\theta}=1$

7.1.5 (From polar to Cartesian coordinates) Show that the system $\dot{r}=r(1-r^2)$, $\dot{\theta}=1$ is equivalent to

$$\dot{x}=x-y-x(x^2+y^2), \qquad \dot{y}=x+y-y(x^2+y^2),$$

where $x=r\cos\theta$, $y=r\sin\theta$. (Hint: $\dot{x}=\frac{d}{dt}(r\cos\theta)=\dot{r}\cos\theta-r\dot{\theta}\sin\theta$.)

7.1.6 (Circuit for van der Pol oscillator) Figure 1 shows the "tetrode multivibrator" circuit used in the earliest commercial radios and analyzed by van der Pol. In van der Pol's day, the active element was a vacuum tube; today it would be a semiconductor device. It acts like an ordinary resistor when I is high, but like a negative resistor (energy source) when I is low. Its current-voltage characteristic $V=f(I)$ resembles a cubic function, as discussed below.

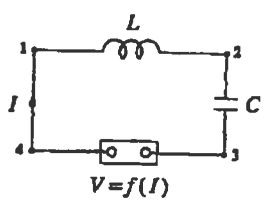

Figure 1

Suppose a source of current is attached to the circuit and then withdrawn. What equations govern the subsequent evolution of the current and the various voltages?

a) Let $V=V_{32}=-V_{23}$ denote the voltage drop from point 3 to point 2 in the circuit. Show that $\dot{V}=-I/C$ and $V=L\dot{I}+f(I)$.

b) Show that the equations in (a) are equivalent to

$$\frac{dw}{d\tau}=-x, \qquad \frac{dx}{d\tau}=w-\mu F(x)$$

where $x=L^{1/2}I$, $w=C^{1/2}V$, $\tau=(LC)^{-1/2}t$, and $F(x)=f(L^{-1/2}x)$.

In Section 7.5, we'll see that this system for (w,x) is equivalent to the van der Pol equation, if $F(x)=\frac{1}{3}x^3-x$. Thus the circuit produces self-sustained oscillations.

7.1.7 (Waveform) Consider the system $\dot{r}=r(4-r^2)$, $\dot{\theta}=1$, and let $x(t)=r(t)\cos\theta(t)$. Given the initial condition $x(0)=0.1$, $y(0)=0$, sketch the approximate waveform of $x(t)$, *without* obtaining an explicit expression for it.

7.1.8 (A circular limit cycle) Consider $\ddot{x}+a\dot{x}(x^2+\dot{x}^2-1)+x=0$, where $a>0$.

a) Find and classify all the fixed points.

b) Show that the system has a circular limit cycle, and find its amplitude and period.

c) Determine the stability of the limit cycle.

d) Give an argument which shows that the limit cycle is unique, i.e., there are no other periodic trajectories.

7.1.9 (Circular pursuit problem) A dog at the center of circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight toward it. In other words, the dog's velocity vector always lies along the line connecting it to the duck. Meanwhile, the duck takes evasive action by swimming around the circumference as fast as it can, always moving counterclockwise.

a) Assuming the pond has unit radius and both animals swim at the same constant speed, derive a pair of differential equations for the path of the dog. (Hint: Use the coordinate system shown in Figure 2 and find equations for $dR/d\theta$ and $d\phi/d\theta$.) Analyze the system. Can you solve it explicitly? Does the dog ever catch the duck?

b) Now suppose the dog swims k times faster than the duck. Derive the differential equations for the dog's path.

c) If $k = \frac{1}{2}$, what does the dog end up doing in the long run?

Note: This problem has a long and intriguing history, dating back to the mid-1800s at least. It is much more difficult than similar *pursuit problems*—there is no known solution for the path of the dog in part (a), in terms of elementary functions. See Davis (1962, pp. 113–125) for a nice analysis and a guide to the literature.

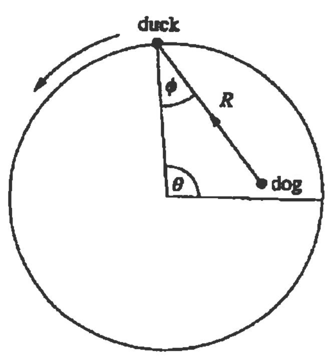

Figure 2

7.2 Ruling Out Closed Orbits

Plot the phase portraits of the following gradient systems $\dot{\mathbf{x}} = -\nabla V$.

7.2.1 $V = x^2 + y^2$ **7.2.2** $V = x^2 - y^2$ **7.2.3** $V = e^x \sin y$

7.2.4 Show that all vector fields on the line are gradient systems. Is the same true of vector fields on the circle?

7.2.5 Let $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$ be a smooth vector field defined on the phase plane.

a) Show that if this is a gradient system, then $\partial f/\partial y = \partial g/\partial x$.

b) Is the condition in (a) also sufficient?

7.2.6 Given that a system is a gradient system, here's how to find its potential function V. Suppose that $\dot{x} = f(x,y)$, $\dot{y} = g(x,y)$. Then $\dot{\mathbf{x}} = -\nabla V$ implies

$f(x,y) = -\partial V/\partial x$ and $g(x,y) = -\partial V/\partial y$. These two equations may be "partially integrated" to find V. Use this procedure to find V for the following gradient systems.

a) $\dot{x} = y^2 + y\cos x, \quad \dot{y} = 2xy + \sin x$

b) $\dot{x} = 3x^2 - 1 - e^{2y}, \quad \dot{y} = -2xe^{2y}$

7.2.7 Consider the system $\dot{x} = y + 2xy$, $\dot{y} = x + x^2 - y^2$.

a) Show that $\partial f/\partial y = \partial g/\partial x$. (Then Exercise 7.2.5(a) implies this is a gradient system.)

b) Find V.

c) Sketch the phase portrait.

7.2.8 Show that the trajectories of a gradient system always cross the equipotentials at right angles (except at fixed points).

7.2.9 For each of the following systems, decide whether it is a gradient system. If so, find V and sketch the phase portrait. On a separate graph, sketch the equipotentials $V = \text{constant}$. (If the system is not a gradient system, go on to the next question.)

a) $\dot{x} = y + x^2 y, \quad \dot{y} = -x + 2xy$

b) $\dot{x} = 2x, \quad \dot{y} = 8y$

c) $\dot{x} = -2xe^{x^2+y^2}, \quad \dot{y} = -2ye^{x^2+y^2}$

7.2.10 Show that the system $\dot{x} = y - x^3$, $\dot{y} = -x - y^3$ has no closed orbits, by constructing a Liapunov function $V = ax^2 + by^2$ with suitable a, b.

7.2.11 Show that $V = ax^2 + 2bxy + cy^2$ is positive definite if and only if $a > 0$ and $ac - b^2 > 0$. (This is a useful criterion that allows us to test for positive definiteness when the quadratic form V includes a "cross term" $2bxy$.)

7.2.12 Show that $\dot{x} = -x + 2y^3 - 2y^4$, $\dot{y} = -x - y + xy$ has no periodic solutions. (Hint: Choose a, m, and n such that $V = x^m + ay^n$ is a Liapunov function.)

7.2.13 Recall the competition model

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2, \qquad \dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2,$$

of Exercise 6.4.6. Using Dulac's criterion with the weighting function $g = (N_1 N_2)^{-1}$, show that the system has no periodic orbits in the first quadrant $N_1, N_2 > 0$.

7.2.14 Consider $\dot{x} = x^2 - y - 1$, $\dot{y} = y(x - 2)$.

a) Show that there are three fixed points and classify them.

b) By considering the three straight lines through pairs of fixed points, show that there are no closed orbits.

c) Sketch the phase portrait.

7.2.15 Consider the system $\dot{x} = x(2-x-y)$, $\dot{y} = y(4x-x^2-3)$. We know from Example 7.2.4 that this system has no closed orbits.

a) Find the three fixed points and classify them.

b) Sketch the phase portrait.

7.2.16 If R is not simply connected, then the conclusion of Dulac's criterion is no longer valid. Find a counterexample.

7.2.17 Assume the hypotheses of Dulac's criterion, except now suppose that R is topologically equivalent to an annulus, i.e., it has exactly one hole in it. Using Green's theorem, show that there exists *at most* one closed orbit in R. (This result can be useful sometimes as a way of proving that a closed orbit is unique.)

7.3 Poincaré–Bendixson Theorem

7.3.1 Consider $\dot{x} = x-y-x(x^2+5y^2)$, $\dot{y} = x+y-y(x^2+y^2)$.

a) Classify the fixed point at the origin.

b) Rewrite the system in polar coordinates, using $r\dot{r} = x\dot{x}+y\dot{y}$ and $\dot{\theta} = (x\dot{y}-y\dot{x})/r^2$.

c) Determine the circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially *outward* component on it.

d) Determine the circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially *inward* component on it.

e) Prove that the system has a limit cycle somewhere in the trapping region $r_1 \le r \le r_2$.

7.3.2 Using numerical integration, compute the limit cycle of Exercise 7.3.1 and verify that it lies in the trapping region you constructed.

7.3.3 Show that the system $\dot{x} = x-y-x^3$, $\dot{y} = x+y-y^3$ has a periodic solution.

7.3.4 Consider the system

$$\dot{x} = x(1-4x^2-y^2)-\tfrac{1}{2}y(1+x), \qquad \dot{y} = y(1-4x^2-y^2)+2x(1+x).$$

a) Show that the origin is an unstable fixed point.

b) By considering $\dot{V}$, where $V = (1-4x^2-y^2)^2$, show that all trajectories approach the ellipse $4x^2+y^2=1$ as $t \to \infty$.

7.3.5 Show that the system $\dot{x} = -x-y+x(x^2+2y^2)$, $\dot{y} = x-y+y(x^2+2y^2)$ has at least one periodic solution.