Solution

$\boxed{6.6.1}$ $\begin{cases} \dot{x} = y(1-x^2) \\ \dot{y} = 1-y^2 \end{cases}$ fixed points $\begin{cases} y(1-x^2)=0 \\ 1-y^2=0 \end{cases}$

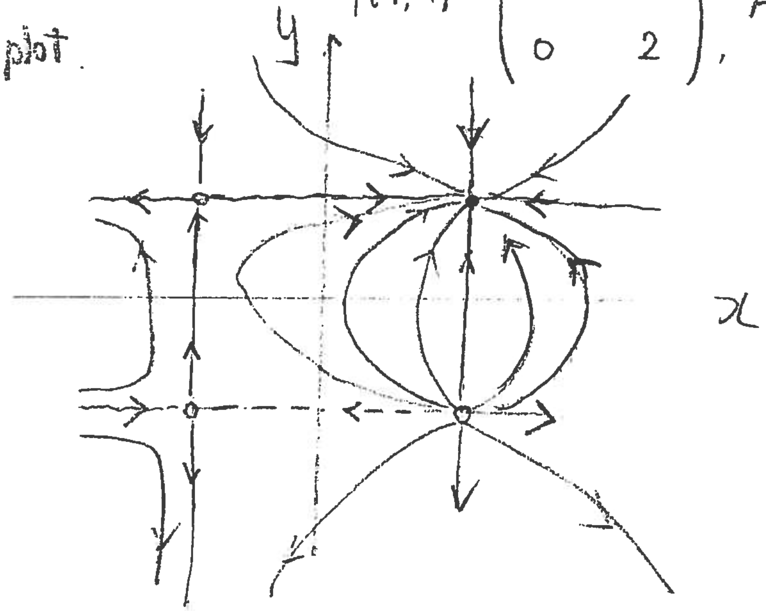
$(x^*, y^*) = (-1,-1),\ (-1,1),\ (1,-1),\ (1,1)$.

Changing the equation by $\begin{cases} t \to -t \\ y \to -y \end{cases}$, we still have the same **equation**. This system is **reversible**.

$A = \begin{pmatrix} -2xy & 1-x^2 \\ 0 & -2y \end{pmatrix}$ $A\big|_{(-1,1)} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$, $A\big|_{(1,1)} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

$A\big|_{(-1,-1)} = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}$, $A\big|_{(1,-1)} = \begin{pmatrix} 2 & 0 \\ 0 & +2 \end{pmatrix}$

The phase plot.



$\boxed{6.6.2}$ $\begin{cases} \dot{x} = y \\ \dot{y} = x\cos y \end{cases}$ fixed points $\begin{cases} y = 0 \\ x\cos y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$

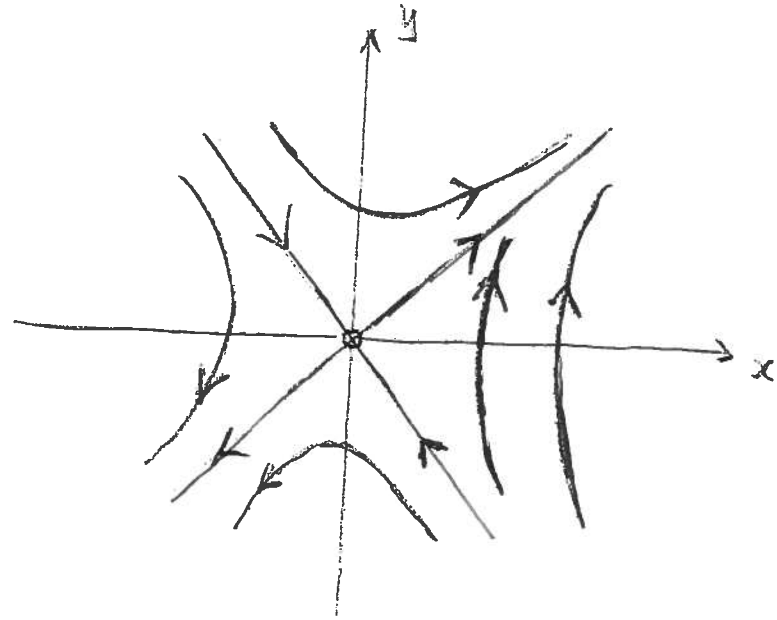
$(t, y) \to (-t, -y)$, we have the same equations. <u>Reversible</u>

$$A = \begin{pmatrix} 0 & 1 \\ \cos y & -x\sin y \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad \lambda_1 = -1, \quad \lambda_2 = 1$$

At $\lambda_1 = -1$ $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. for any c

At $\lambda_2 = 1$ $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



$\boxed{6.6.3}$ $\begin{cases} \dot{x} = \sin y \\ \dot{y} = \sin x \end{cases}$

a). $t \to -t$, $y \to -y$ we have the same system. Reversible

b) $\begin{cases} \sin y = 0 \\ \sin x = 0 \end{cases}$ $\quad y = n\pi$, $x = m\pi$ for any $m, n \in \mathbb{Z}$

c). $\begin{cases} \frac{dx}{dt} = \sin y \\ \frac{dy}{dt} = \sin x \end{cases} \Rightarrow \frac{dy}{dx} = \frac{\sin x}{\sin y} \Rightarrow \sin y \, dy = \sin x \, dx$

$$\int \sin y \, dy = \int \sin x \, dx$$

$$-\cos y = -\cos x + C$$

$$\cos y(t) = \cos x(t) + C.$$

Assume $\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases} \Rightarrow C = \cos y_0 - \cos x_0$. e.g.,

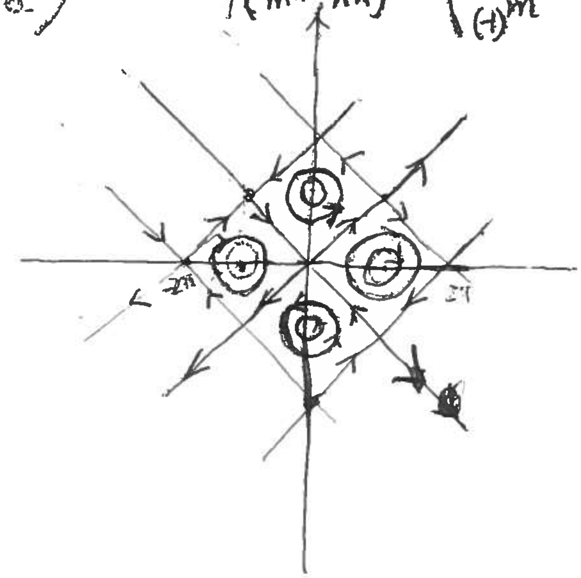
$$\cos y(t) - \cos x(t) = \cos y_0 - \cos x_0.$$

If initially $y_0 = \pm x_0$, then $\cos y - \cos x = 0$.

$\Rightarrow y = \pm x$ as the solution

d). $A = \begin{pmatrix} 0 & \cos y \\ \cos x & 0 \end{pmatrix}$ $\quad A\Big|_{(m\pi, n\pi)} = \begin{pmatrix} 0 & (-1)^n \\ (-1)^m & 0 \end{pmatrix}$.

phase plot.

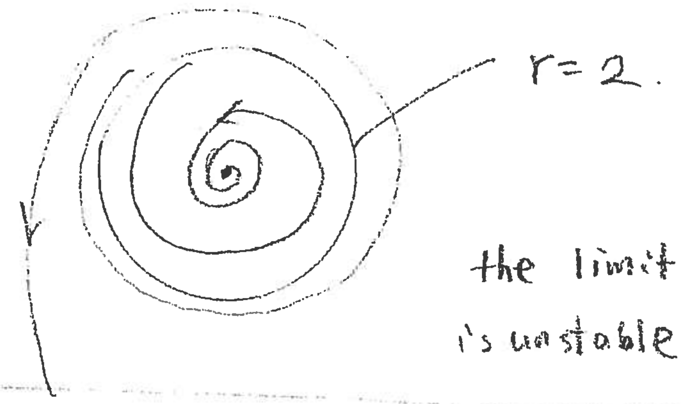


$\boxed{7.1.1}$ $\begin{cases} \dot{r} = r^3 - 4r \\ \theta' = 1 \end{cases}$

for the first equation: $r^3 - 4r = 0 \Rightarrow r = 0$ or $r^2 = 4$

r is positive $\Rightarrow$ $r = 2$

$\dot{r} = r(r-2)(r+2) = f(r)$ $\begin{cases} f(r) > 0 \text{ if } r > 2 \\ f(r) < 0 \text{ if } 0 < r < 2 \end{cases}$



the limit cycle $r = 2$ is unstable.

$\boxed{7.1.3}$ $\begin{cases} \dot{r} = f(r) = r(1-r^2)(4-r^2) \\ \theta' = 2 - r^2 \end{cases}$

$f(r) = 0 \Rightarrow r_1 = 0$, $r_2 = 1$, $r_3 = 2$.

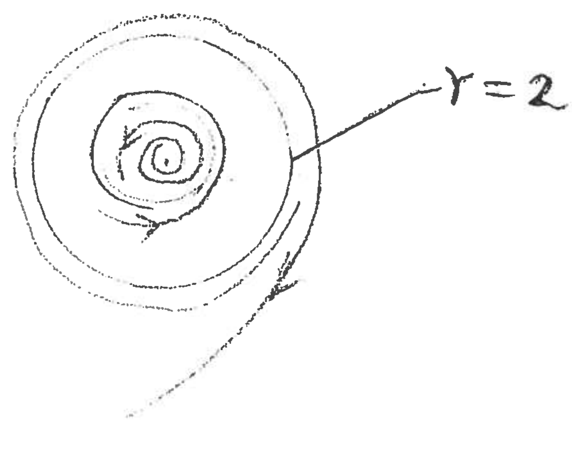
$f(r) > 0$ if $r > 2$

$f(r) < 0$ if $1 < r < 2$

$f(r) > 0$ if $0 < r < 1$

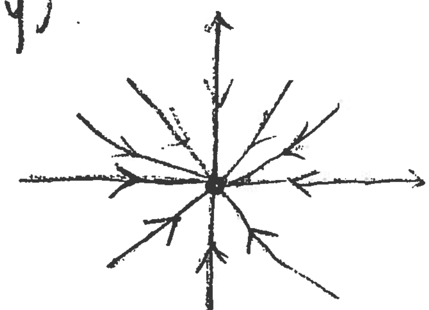
$r = 2$ is unstable

$r = 1$ is stable.

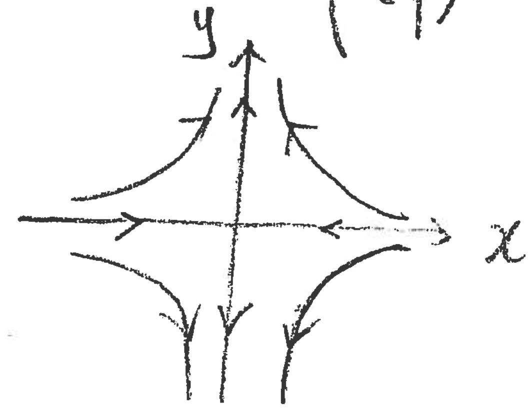


7.2.1: $\dot{x} = -\nabla V$. $\quad V = x^2+y^2 \quad -\nabla V = \begin{pmatrix} -2x \\ -2y \end{pmatrix}$

$\dot{x} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.



7.2.2 $\quad V = x^2 - y^2 \quad -\nabla V = \begin{pmatrix} -2x \\ 2y \end{pmatrix}$. $\quad \dot{x} = \begin{pmatrix} -2 & \\ & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$



phase plot

7.2.7 $\quad \begin{cases} \dot{x} = y + 2xy = f \\ \dot{y} = x + x^2 - y^2 = g \end{cases}$

a). $\frac{\partial f}{\partial y} = 1+2x$, $\quad \frac{\partial g}{\partial x} = 1+2x$. $\quad \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x}$.

b). $\frac{\partial V}{\partial x} = -f = -(y+2xy) \Rightarrow V = -(xy + x^2y) + h(y)$

for any function $h(y)$

$\frac{\partial V}{\partial y} = -g$; $\quad -x - x^2 + h'(y) = -g = -x - x^2 + y^2$

$\Rightarrow h'(y) = y^2 \Rightarrow h(y) = \frac{1}{3}y^3 + c$.

$V = -(xy + x^2y) + \frac{1}{3}y^3 + c$.

7.27.

a) $\begin{cases} y(1+2x)=0 \\ x+x^2-y^2=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=-1 \\ y=0 \end{cases}$

$A=\begin{pmatrix} 2y & 1+2x \\ 1+2x & -2y \end{pmatrix}$

$A\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. $\quad \lambda_{1,2} = \pm 1$

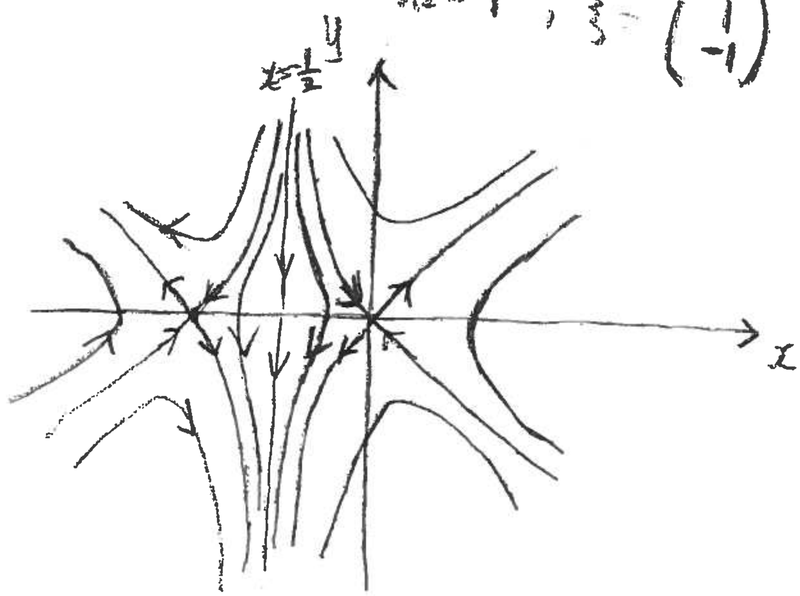
$\lambda_1=-1$. $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2=1$. $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A\big|_{(-1,0)} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \lambda_{1,2} = \pm 1$.

$\lambda_1 = -1$, $\xi = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$, $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.



7.2.10

$\dot{x} = y - x^3 \quad \dot{y} = -x - y^3$

$V = ax^2 + by^2$

$\frac{dV}{dt} = 2axx' + 2byy' = 2ax(y - x^3) + 2by(-x - y^3)$

$= -2ax^4 + 2axy + 2bxy \cdot (-1) - 2by^4$

choose $a = b = 1$

$\frac{dV}{dt} = -2x^4 - 2y^4 < 0$ if $(x,y) \neq (0,0)$.

$V = x^2 + y^2 > 0$ if $(x,y) \neq (0,0)$

Therefore V is the Liapunov function of the system

$(0,0)$ is globally stable

$\boxed{7.3.1}$ $\dot{x} = x - y - x(x^2 + 5y^2)$, $y' = x + y - y(x^2 + y^2)$.

a) $\begin{cases} x - y - x(x^2 + 5y^2) = 0 \\ x + y - y(x^2 + y^2) = 0 \end{cases}$ $\quad (x,y) = (0,0)$ is a fixed point

$A = \begin{pmatrix} 1 - 3x^2 - 5y^2 & -1 - 10xy \\ 1 - 2xy & 1 - 10x^2 - 3y^2 \end{pmatrix}$ $\quad A\Big|_{(0,0)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$(1-\lambda)^2 + 1 = 0 \quad 1 - \lambda = \pm i \quad \lambda = 1 \pm i$. $\mathrm{Re}(\lambda) > 0$

$(0,0)$ is an unstable spiral.

b) $r' = \frac{x'x + y'y}{r} = \frac{1}{r}\left[x(x - y - x(x^2 + 5y^2)) + y(x + y - y(x^2 + y^2))\right]$

$= \frac{1}{r}\left[x^2 + y^2 - x^2(x^2 + 5y^2) - y^2(x^2 + y^2)\right]$

$= r - r^3\left[\cos^2\theta\,(\cos^2\theta + 5\sin^2\theta) + \sin^2\theta\right]$

$$\theta' = \frac{xy' - yx'}{r^2} = 1 + 4r^2 \cos\theta \sin^3\theta$$

c) $r' = r(1 - r^2 f(\theta))$, $f(\theta) = \cos^2\theta\,[\cos^2\theta + 5\sin^2\theta] + \sin^2\theta$

$= r\,(1 + r\sqrt{f(\theta)})\,(1 - r\sqrt{f(\theta)})$

$= r\,(1 + r\sqrt{f(\theta)})\,(\frac{1}{\sqrt{f(\theta)}} - r) \cdot \sqrt{f(\theta)}$

set

$t = \cos^2\theta$, $0 \le t \le 1$

$f(\theta) = t\,[t + 5(1-t)] + (1-t)$

$= 1 + 4t - 4t^2$. $t \in [0,1]$

$f|_{t=\frac{1}{2}} = 2$ $\quad f|_{t=0} = 1$ $\quad f|_{t=1} = 1$

$1 \le f \le 2$

$\frac{1}{\sqrt{2}} \le \frac{1}{\sqrt{f(\theta)}} \le \frac{1}{\sqrt{1}} = 1$

choose $r_1 \le \frac{1}{\sqrt{2}}$ then

$r'|_{r=r_1} = r(1 + r\sqrt{f(\theta)})\,(\frac{1}{\sqrt{f(\theta)}} - r) \cdot \sqrt{f(\theta)}\,\Big|_{r=r_1}$

≥ 0

every orbit on $r = r_1$ is radially outward

d) choose $r_2 \geq 1$ $\quad r'|_{r=r_2} \le 0$

every orbit on $r = r_2$ is radially inward

e) By Poincare-Bendison theorem, there is a limit cycle in $r_1 \le r \le r_2$