has one fixed point, a saddle at $(-1,0)$. Its unstable manifold is the x-axis, but its stable manifold is a curve that is harder to find. The goal of this exercise is to approximate this unknown curve.

a) Let (x,y) be a point on the stable manifold, and assume that (x,y) is close to $(-1,0)$. Introduce a new variable $u=x+1$, and write the stable manifold as $y=a_1u+a_2u^2+O(u^3)$. To determine the coefficients, derive two expressions for dy/du and equate them.
b) Check that your analytical result produces a curve with the same shape as the stable manifold shown in Figure 6.1.4.

6.2 Existence, Uniqueness, and Topological Consequences

6.2.1 We claimed that different trajectories can never intersect. But in many phase portraits, different trajectories appear to intersect at a fixed point. Is there a contradiction here?

6.2.2 Consider the system $\dot{x}=y$, $\dot{y}=-x+(1-x^2-y^2)y$.

a) Let D be the open disk $x^2+y^2<4$. Verify that the system satisfies the hypotheses of the existence and uniqueness theorem throughout the domain D.
b) By substitution, show that $x(t)=\sin t$, $y(t)=\cos t$ is an exact solution of the system.
c) Now consider a different solution, in this case starting from the initial condition $x(0)=\frac{1}{2}$, $y(0)=0$. Without doing any calculations, explain why this solution *must* satisfy $x(t)^2+y(t)^2<1$ for all $t<\infty$.

6.3 Fixed Points and Linearization

For each of the following systems, find the fixed points, classify them, sketch the neighboring trajectories, and try to fill in the rest of the phase portrait.

✓ **6.3.1** $\dot{x}=x-y$, $\dot{y}=x^2-4$

6.3.2 $\dot{x}=\sin y$, $\dot{y}=x-x^3$

✓ **6.3.3** $\dot{x}=1+y-e^{-x}$, $\dot{y}=x^3-y$

6.3.4 $\dot{x}=y+x-x^3$, $\dot{y}=-y$

✓ **6.3.5** $\dot{x}=\sin y$, $\dot{y}=\cos x$

6.3.6 $\dot{x}=xy-1$, $\dot{y}=x-y^3$

6.3.7 For each of the nonlinear systems above, plot a computer-generated phase portrait and compare to your approximate sketch.

6.3.8 (Gravitational equilibrium) A particle moves along a line joining two stationary masses, m_1 and m_2, which are separated by a fixed distance a. Let x denote the distance of the particle from m_1.

a) Show that $\ddot{x}=\dfrac{Gm_2}{(x-a)^2}-\dfrac{Gm_1}{x^2}$, where G is the gravitational constant.
b) Find the particle's equilibrium position. Is it stable or unstable?

6.3.9 Consider the system $\dot{x} = y^3 - 4x$, $\dot{y} = y^3 - y - 3x$.

a) Find all the fixed points and classify them.

b) Show that the line $x = y$ is invariant, i.e., any trajectory that starts on it stays on it.

c) Show that $|x(t) - y(t)| \to 0$ as $t \to \infty$ for all other trajectories. (Hint: Form a differential equation for $x - y$.)

d) Sketch the phase portrait.

e) If you have access to a computer, plot an accurate phase portrait on the square domain $-20 \le x, y \le 20$. (To avoid numerical instability, you'll need to use a fairly small step size, because of the strong cubic nonlinearity.) Notice the trajectories seem to approach a certain curve as $t \to -\infty$; can you explain this behavior intuitively, and perhaps find an approximate equation for this curve?

6.3.10 (Dealing with a fixed point for which linearization is inconclusive) The goal of this exercise is to sketch the phase portrait for $\dot{x} = xy$, $\dot{y} = x^2 - y$.

a) Show that the linearization predicts that the origin is a non-isolated fixed point.

b) Show that the origin is in fact an isolated fixed point.

c) Is the origin repelling, attracting, a saddle, or what? Sketch the vector field along the nullclines and at other points in the phase plane. Use this information to sketch the phase portrait.

d) Plot a computer-generated phase portrait to check your answer to (c).

(Note: This problem can also be solved by a method called *center manifold theory*, as explained in Wiggins (1990) and Guckenheimer and Holmes (1983).)

6.3.11 (Nonlinear terms can change a star into a spiral) Here's another example that shows that borderline fixed points are sensitive to nonlinear terms. Consider the system in polar coordinates given by $\dot{r} = -r$, $\dot{\theta} = 1/\ln r$.

a) Find $r(t)$ and $\theta(t)$ explicitly, given an initial condition (r_0, θ_0).

b) Show that $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \infty$. Therefore the origin is a stable spiral for the nonlinear system.

c) Write the system in x, y coordinates.

d) Show that the linearized system about the origin is $\dot{x} = -x$, $\dot{y} = -y$. Thus the origin is a stable star for the linearized system.

6.3.12 (Polar coordinates) Using the identity $\theta = \tan^{-1}(y/x)$, show that $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

6.3.13 (Another linear center that's actually a nonlinear spiral) Consider the system $\dot{x} = -y - x^3$, $\dot{y} = x$. Show that the origin is a spiral, although the linearization predicts a center.

√ **6.3.14** Classify the fixed point at the origin for the system $\dot{x} = -y + ax^3$, $\dot{y} = x + ay^3$, for all real values of the parameter a.

6.3.15 Consider the system $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1-\cos\theta$, where r, θ represent polar coordinates. Sketch the phase portrait and thereby show that the fixed point $r^* = 1$, $\theta^* = 0$ is attracting but not Liapunov stable.

6.3.16 (Saddle switching and structural stability) Consider the system $\dot{x} = a + x^2 - xy$, $\dot{y} = y^2 - x^2 - 1$, where a is a parameter.

a) Sketch the phase portrait for $a = 0$. Show that there is a trajectory connecting two saddle points. (Such a trajectory is called a *saddle connection.*)
b) With the aid of a computer if necessary, sketch the phase portrait for $a < 0$ and $a > 0$.

Notice that for $a \neq 0$, the phase portrait has a different topological character: the saddles are no longer connected by a trajectory. The point of this exercise is that the phase portrait in (a) is *not structurally stable*, since its topology can be changed by an arbitrarily small perturbation a.

✓ **6.3.17** (Nasty fixed point) The system $\dot{x} = xy - x^2y + y^3$, $\dot{y} = y^2 + x^3 - xy^2$ has a nasty higher-order fixed point at the origin. Using polar coordinates or otherwise, sketch the phase portrait.

6.4 Rabbits versus Sheep

Consider the following "rabbits vs. sheep" problems, where $x, y \geq 0$. Find the fixed points, investigate their stability, draw the nullclines, and sketch plausible phase portraits. Indicate the basins of attraction of any stable fixed points.

✓ **6.4.1** $\dot{x} = x(3-x-y)$, $\dot{y} = y(2-x-y)$

6.4.2 $\dot{x} = x(3-2x-y)$, $\dot{y} = y(2-x-y)$

✓ **6.4.3** $\dot{x} = x(3-2x-2y)$, $\dot{y} = y(2-x-y)$

The next three exercises deal with competition models of increasing complexity. We assume $N_1, N_2 \geq 0$ in all cases.

6.4.4 The simplest model is $\dot{N}_1 = r_1N_1 - b_1N_1N_2$, $\dot{N}_2 = r_2N_2 - b_2N_1N_2$.

a) In what way is this model less realistic than the one considered in the text?
b) Show that by suitable rescalings of N_1, N_2, and t, the model can be nondimensionalized to $x' = x(1-y)$, $y' = y(\rho - x)$. Find a formula for the dimensionless group ρ.
c) Sketch the nullclines and vector field for the system in (b).
d) Draw the phase portrait, and comment on the biological implications.
e) Show that (almost) all trajectories are curves of the form $\rho \ln x - x = \ln y - y + C$. (Hint: Derive a differential equation for dx/dy, and separate the variables.) Which trajectories are not of the stated form?

6.4.5 Now suppose that species #1 has a finite carrying capacity K_1. Thus

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 - b_2 N_1 N_2 .$$

Nondimensionalize the model and analyze it. Show that there are two qualitatively different kinds of phase portrait, depending on the size of K_1. (Hint: Draw the nullclines.) Describe the long-term behavior in each case.

6.4.6 Finally, suppose that both species have finite carrying capacities:

$$\dot{N}_1 = r_1 N_1(1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2(1 - N_2/K_2) - b_2 N_1 N_2 .$$

a) Nondimensionalize the model. How many dimensionless groups are needed?
b) Show that there are four qualitatively different phase portraits, as far as long-term behavior is concerned.
c) Find conditions under which the two species can stably coexist. Explain the biological meaning of these conditions. (Hint: The carrying capacities reflect the competition *within* a species, whereas the b's reflect the competition *between* species.)

6.4.7 (Two-mode laser) According to Haken (1983, p. 129), a two-mode laser produces two different kinds of photons with numbers n_1 and n_2. By analogy with the simple laser model discussed in Section 3.3, the rate equations are

$$\dot{n}_1 = G_1 N n_1 - k_1 n_1$$
$$\dot{n}_2 = G_2 N n_2 - k_2 n_2$$

where $N(t) = N_0 - \alpha_1 n_1 - \alpha_2 n_2$ is the number of excited atoms. The parameters $G_1, G_2, k_1, k_2, \alpha_1, \alpha_2, N_0$ are all positive.
a) Discuss the stability of the fixed point $n_1{}^* = n_2{}^* = 0$.
b) Find and classify any other fixed points that may exist.
c) Depending on the values of the various parameters, how many qualitatively different phase portraits can occur? For each case, what does the model predict about the long-term behavior of the laser?

6.5 Conservative Systems

6.5.1 Consider the system $\ddot{x} = x^3 - x$.
a) Find all the equilibrium points and classify them.
b) Find a conserved quantity.
c) Sketch the phase portrait.

√ **6.5.2** Consider the system $\ddot{x} = x - x^2$.
a) Find and classify the equilibrium points.

b) Sketch the phase portrait.
c) Find an equation for the homoclinic orbit that separates closed and nonclosed trajectories.

6.5.3 Find a conserved quantity for the system $\ddot{x} = a - e^x$, and sketch the phase portrait for $a < 0$, $a = 0$, and $a > 0$.

6.5.4 Sketch the phase portrait for the system $\ddot{x} = ax - x^2$ for $a < 0$, $a = 0$, and $a > 0$.

6.5.5 Investigate the stability of the equilibrium points of the system $\ddot{x} = (x-a)(x^2-a)$ for all real values of the parameter a. (Hints: It might help to graph the right-hand side. An alternative is to rewrite the equation as $\ddot{x} = -V'(x)$ for a suitable potential energy function V and then use your intuition about particles moving in potentials.)

6.5.6 (Epidemic model revisited) In Exercise 3.7.6, you analyzed the Kermack–McKendrick model of an epidemic by reducing it to a certain first-order system. In this problem you'll see how much easier the analysis becomes in the phase plane. As before, let $x(t) \geq 0$ denote the size of the healthy population and $y(t) \geq 0$ denote the size of the sick population. Then the model is

$$\dot{x} = -kxy, \qquad \dot{y} = kxy - \ell y$$

where $k, \ell > 0$. (The equation for $z(t)$, the number of deaths, plays no role in the x, y dynamics so we omit it.)
a) Find and classify all the fixed points.
b) Sketch the nullclines and the vector field.
c) Find a conserved quantity for the system. (Hint: Form a differential equation for dy/dx. Separate the variables and integrate both sides.)
d) Plot the phase portrait. What happens as $t \to \infty$?
e) Let (x_0, y_0) be the initial condition. An *epidemic* is said to occur if $y(t)$ increases initially. Under what condition does an epidemic occur?

6.5.7 (General relativity and planetary orbits) The relativistic equation for the orbit of a planet around the sun is

$$\frac{d^2u}{d\theta^2} + u = \alpha + \varepsilon u^2$$

where $u = 1/r$ and r, θ are the polar coordinates of the planet in its plane of motion. The parameter α is positive and can be found explicitly from classical Newtonian mechanics; the term εu^2 is Einstein's correction. Here ε is a very small positive parameter.
a) Rewrite the equation as a system in the (u, v) phase plane, where $v = du/d\theta$.

b) Find all the equilibrium points of the system.
c) Show that one of the equilibria is a center in the (u, v) phase plane, according to the linearization. Is it a *nonlinear* center?
d) Show that the equilibrium point found in (c) corresponds to a circular planetary orbit.

Hamiltonian systems are fundamental to classical mechanics; they provide an equivalent but more geometric version of Newton's laws. They are also central to celestial mechanics and plasma physics, where dissipation can sometimes be neglected on the time scales of interest. The theory of Hamiltonian systems is deep and beautiful, but perhaps too specialized and subtle for a first course on nonlinear dynamics. See Arnold (1978), Lichtenberg and Lieberman (1992), Tabor (1989), or Hénon (1983) for introductions.

Here's the simplest instance of a Hamiltonian system. Let $H(p,q)$ be a smooth, real-valued function of two variables. The variable q is the "generalized coordinate" and p is the "conjugate momentum." (In some physical settings, H could also depend explicitly on time t, but we'll ignore that possibility.) Then a system of the form

$$\dot{q} = \partial H/\partial p, \qquad \dot{p} = -\partial H/\partial q$$

is called a ***Hamiltonian system*** and the function H is called the ***Hamiltonian***. The equations for $\dot{q}$ and $\dot{p}$ are called Hamilton's equations.

The next three exercises concern Hamiltonian systems.

6.5.8 (Harmonic oscillator) For a simple harmonic oscillator of mass m, spring constant k, displacement x, and momentum p, the Hamiltonian is $H = \frac{p^2}{2m} + \frac{kx^2}{2}$. Write out Hamilton's equations explicitly. Show that one equation gives the usual definition of momentum and the other is equivalent to $F = ma$. Verify that H is the total energy.

✓ **6.5.9** Show that for any Hamiltonian system, $H(x,p)$ is a conserved quantity. (Hint: Show $\dot{H} = 0$ by applying the chain rule and invoking Hamilton's equations.) Hence the trajectories lie on the contour curves $H(x,p) = C$.

6.5.10 (Inverse-square law) A particle moves in a plane under the influence of an inverse-square force. It is governed by the Hamiltonian $H(p,r) = \frac{p^2}{2} + \frac{h^2}{2r^2} - \frac{k}{r}$ where $r > 0$ is the distance from the origin and p is the radial momentum. The parameters h and k are the angular momentum and the force constant, respectively.
a) Suppose $k > 0$, corresponding to an attractive force like gravity. Sketch the