shown in Figure 3.7.5. (Exercise 3.7.2 deals with some of the analytical properti
of these curves.)

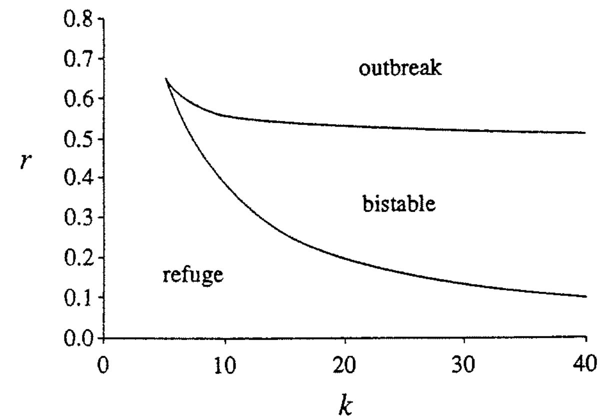

Figure 3.7.5

The different regions in Figure 3.7.5 are labeled according to the stable fixe points that exist. The refuge level a is the only stable state for low r, and the out break level c is the only stable state for large r. In the ***bistable*** region, both stabl states exist.

The stability diagram is very similar to Figure 3.6.2. It too can be regarded a the projection of a cusp catastrophe surface, as schematically illustrated in Figur 3.7.6. You are hereby challenged to graph the surface accurately!

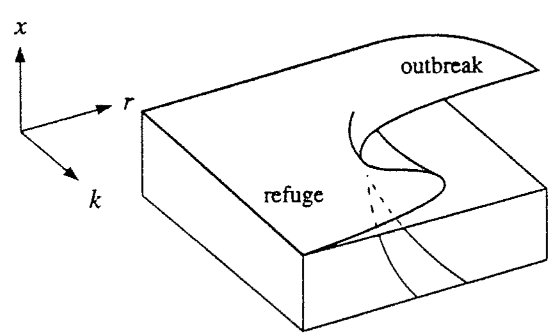

Figure 3.7.6

Comparison with Observations

Now we need to decide on biologically plausible values of the dimensionless groups $r = RA/B$ and $k = K/A$. A complication is that these parameters may drift

They reason as follows: let S denote the average size of the trees, interpreted as the total surface area of the branches in a stand. Then the carrying capacity K should be proportional to the available foliage, so $K = K'S$. Similarly, the half-saturation parameter A in the predation term should be proportional to S; predators such as birds search *units of foliage*, not acres of forest, and so the relevant quantity A' must have the dimensions of budworms per unit of branch area. Hence $A = A'S$ and therefore

$$r = \frac{RA'}{B}S, \qquad k = \frac{K'}{A'}. \tag{10}$$

The experimental observations suggest that for a young forest, typically $k \approx 300$ and $r < 1/2$ so the parameters lie in the bistable region. The budworm population is kept down by the birds, which find it easy to search the small number of branches per acre. However, as the forest grows, S increases and therefore the point (k, r) drifts upward in parameter space toward the outbreak region of Figure 3.7.5. Ludwig et al. (1978) estimate that $r \approx 1$ for a fully mature forest, which lies dangerously in the outbreak region. After an outbreak occurs, the fir trees die and the forest is taken over by birch trees. But they are less efficient at using nutrients and eventually the fir trees come back—this recovery takes about 50–100 years (Murray 1989).

We conclude by mentioning some of the approximations in the model presented here. The tree dynamics have been neglected; see Ludwig et al. (1978) for a discussion of this longer time-scale behavior. We've also neglected the *spatial* distribution of budworms and their possible dispersal—see Ludwig et al. (1979) and Murray (1989) for treatments of this aspect of the problem.

EXERCISES FOR CHAPTER 3

3.1 Saddle-Node Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a saddle-node bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

√**3.1.1** $\dot{x} = 1 + rx + x^2$ **3.1.2** $\dot{x} = r - \cosh x$

√**3.1.3** $\dot{x} = r + x - \ln(1 + x)$ **3.1.4** $\dot{x} = r + \frac{1}{2}x - x/(1 + x)$

√**3.1.5** (Unusual bifurcations) In discussing the normal form of the saddle-node bifurcation, we mentioned the assumption that $a = \partial f/\partial r|_{(x^*,r_c)} \neq 0$. To see what can happen if $\partial f/\partial r|_{(x^*,r_c)} = 0$, sketch the vector fields for the following examples, and then plot the fixed points as a function of r.

a) $\dot{x} = r^2 - x^2$
b) $\dot{x} = r^2 + x^2$

3.2 Transcritical Bifurcation

For each of the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a transcritical bifurcation occurs at a critical value of r, to be determined. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

√ **3.2.1** $\dot{x} = rx + x^2$ **3.2.2** $\dot{x} = rx - \ln(1+x)$

√ **3.2.3** $\dot{x} = x - rx(1-x)$ **3.2.4** $\dot{x} = x(r - e^x)$

3.2.5 (Chemical kinetics) Consider the chemical reaction system

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X \qquad X + B \xrightarrow{k_2} C.$$

This is a generalization of Exercise 2.3.2; the new feature is that X is used up in the production of C.

a) Assuming that both A and B are kept at constant concentrations a and b, show that the law of mass action leads to an equation of the form $\dot{x} = c_1 x - c_2 x^2$, where x is the concentration of X, and c_1 and c_2 are constants to be determined.
b) Show that $x^* = 0$ is stable when $k_2 b > k_1 a$, and explain why this makes sense chemically.

The next two exercises concern the normal form for the transcritical bifurcation. In Example 3.2.2, we showed how to reduce the dynamics near a transcritical bifurcation to the approximate form $\dot{X} = RX - X^2 + O(X^3)$. Our goal now is to show that the $O(X^3)$ terms can always be eliminated by a suitable nonlinear change of variables; in other words, the reduction to normal form can be made *exact*, not just approximate.

3.2.6 (Eliminating the cubic term) Consider the system $\dot{X} = RX - X^2 + aX^3 + O(X^4)$, where $R \neq 0$. We want to find a new variable x such that the system transforms into $\dot{x} = Rx - x^2 + O(x^4)$. This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped up to fourth order.

Let $x = X + bX^3 + O(X^4)$, where b will be chosen later to eliminate the cubic term in the differential equation for x. This is called a *near-identity transformation*, since x and X are practically equal; they differ by a tiny cubic term. (We have skipped the quadratic term X^2, because it is not needed—you should check this later.) Now we need to rewrite the system in terms of x; this calculation requires a few steps.

a) Show that the near-identity transformation can be inverted to yield $X = x + cx^3 + O(x^4)$, and solve for c.
b) Write $\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)$, and substitute for X and $\dot{X}$ on the right-hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\dot{x} = Rx - x^2 + kx^3 + O(x^4)$, where k depends on a, b, and R.
c) Now the moment of triumph: Choose b so that $k = 0$.
d) Is is really necessary to make the assumption that $R \neq 0$? Explain.

3.2.7 (Eliminating any higher-order term) Now we generalize the method of the last exercise. Suppose we have managed to eliminate a number of higher-order terms, so that the system has been transformed into $\dot{X} = RX - X^2 + a_n X^n + O(X^{n+1})$, where $n \geq 3$. Use the near-identity transformation $x = X + b_n X^n + O(X^{n+1})$ and the previous strategy to show that the system can be rewritten as $\dot{x} = Rx - x^2 + O(x^{n+1})$ for an appropriate choice of b_n. Thus we can eliminate as many higher-order terms as we like.

3.3 Laser Threshold

3.3.1 (An improved model of a laser) In the simple laser model considered in Section 3.3, we wrote an *algebraic* equation relating N, the number of excited atoms, to n, the number of laser photons. In more realistic models, this would be replaced by a *differential* equation. For instance, Milonni and Eberly (1988) show that after certain reasonable approximations, quantum mechanics leads to the system

$$\dot{n} = GnN - kn$$
$$\dot{N} = -GnN - fN + p.$$

Here G is the gain coefficient for stimulated emission, k is the decay rate due to loss of photons by mirror transmission, scattering, etc., f is the decay rate for spontaneous emission, and p is the pump strength. All parameters are positive, except p, which can have either sign.

This two-dimensional system will be analyzed in Exercise 8.1.13. For now, let's convert it to a one-dimensional system, as follows.

a) Suppose that N relaxes much more rapidly than n. Then we may make the quasi-static approximation $\dot{N} \approx 0$. Given this approximation, express $N(t)$ in terms of $n(t)$ and derive a first-order system for n. (This procedure is often called ***adiabatic climination***, and one says that the evolution of $N(t)$ is *slaved* to that of $n(t)$. See Haken (1983).)
b) Show that $n^* = 0$ becomes unstable for $p > p_c$, where p_c is to be determined.

c) What type of bifurcation occurs at the laser threshold p_c?
d) (Hard question) For what range of parameters is it valid to make the approximation used in (a)?

3.3.2 (Maxwell–Bloch equations) The Maxwell–Bloch equations provide an even more sophisticated model for a laser. These equations describe the dynamics of the electric field E, the mean polarization P of the atoms, and the population inversion D:

$$\dot{E} = \kappa(P - E)$$
$$\dot{P} = \gamma_1(ED - P)$$
$$\dot{D} = \gamma_2(\lambda + 1 - D - \lambda EP)$$

where κ is the decay rate in the laser cavity due to beam transmission, γ_1 and γ_2 are decay rates of the atomic polarization and population inversion, respectively, and λ is a pumping energy parameter. The parameter λ may be positive, negative, or zero; all the other parameters are positive.

These equations are similar to the Lorenz equations and can exhibit chaotic behavior (Haken 1983, Weiss and Vilaseca 1991). However, many practical lasers do not operate in the chaotic regime. In the simplest case $\gamma_1, \gamma_2 >> \kappa$; then P and D relax rapidly to steady values, and hence may be adiabatically eliminated, as follows.

a) Assuming $\dot{P} \approx 0$, $\dot{D} \approx 0$, express P and D in terms of E, and thereby derive a first-order equation for the evolution of E.
b) Find all the fixed points of the equation for E.
c) Draw the bifurcation diagram of E^* vs. λ. (Be sure to distinguish between stable and unstable branches.)

3.4 Pitchfork Bifurcation

In the following exercises, sketch all the qualitatively different vector fields that occur as r is varied. Show that a pitchfork bifurcation occurs at a critical value of r (to be determined) and classify the bifurcation as supercritical or subcritical. Finally, sketch the bifurcation diagram of x^* vs. r.

√ **3.4.1** $\dot{x} = rx + 4x^3$ **3.4.2** $\dot{x} = rx - \sinh x$

√ **3.4.3** $\dot{x} = rx - 4x^3$ √**3.4.4** $\dot{x} = x + \frac{rx}{1+x^2}$

The next exercises are designed to test your ability to distinguish among the various types of bifurcations—it's easy to confuse them! In each case, find the values of r at which bifurcations occur, and classify those as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* vs. r.

√**3.4.5** $\dot{x} = r - 3x^2$ **3.4.6** $\dot{x} = rx - \frac{x}{1+x}$

√**3.4.7** $\dot{x} = 5 - re^{-x^2}$ **3.4.8** $\dot{x} = rx - \frac{x}{1+x^2}$

√**3.4.9** $\dot{x} = x + \tanh(rx)$ **3.4.10** $\dot{x} = rx + \frac{x^3}{1+x^2}$

3.4.11 (An interesting bifurcation diagram) Consider the system $\dot{x} = rx - \sin x$.
a) For the case $r = 0$, find and classify all the fixed points, and sketch the vector field.
b) Show that when $r > 1$, there is only one fixed point. What kind of fixed point is it?
c) As r decreases from ∞ to 0, classify *all* the bifurcations that occur.
d) For $0 < r << 1$, find an approximate formula for values of r at which bifurcations occur.
e) Now classify all the bifurcations that occur as r decreases from 0 to $-\infty$.
f) Plot the bifurcation diagram for $-\infty < r < \infty$, and indicate the stability of the various branches of fixed points.

3.4.12 ("Quadfurcation") With tongue in cheek, we pointed out that the pitchfork bifurcation could be called a "trifurcation," since three branches of fixed points appear for $r > 0$. Can you construct an example of a "quadfurcation," in which $\dot{x} = f(x, r)$ has no fixed points for $r < 0$ and four branches of fixed points for $r > 0$? Extend your results to the case of an arbitrary number of branches, if possible.

3.4.13 (Computer work on bifurcation diagrams) For the vector fields below, use a computer to obtain a quantitatively accurate plot of the values of x^* vs. r, where $0 \le r \le 3$. In each case, there's an easy way to do this, and a harder way using the Newton-Raphson method.

a) $\dot{x} = r - x - e^{-x}$ b) $\dot{x} = 1 - x - e^{-rx}$

3.4.14 (Subcritical pitchfork) Consider the system $\dot{x} = rx + x^3 - x^5$, which exhibits a subcritical pitchfork bifurcation.
a) Find algebraic expressions for all the fixed points as r varies.
b) Sketch the vector fields as r varies. Be sure to indicate all the fixed points and their stability.
c) Calculate r_s, the parameter value at which the nonzero fixed points are born in a saddle-node bifurcation.

3.4.15 (First-order phase transition) Consider the potential $V(x)$ for the system $\dot{x} = rx + x^3 - x^5$. Calculate r_c, where r_c is defined by the condition that V has three equally deep wells, i.e., the values of V at the three local minima are equal.

(Note: In equilibrium statistical mechanics, one says that a *first-order pha* *transition* occurs at $r = r_c$. For this value of r, there is equal probability of findi the system in the state corresponding to any of the three minima. The freezing water into ice is the most familiar example of a first-order phase transition.)

3.4.16 (Potentials) In parts (a)–(c), let $V(x)$ be the potential, in the sense th $\dot{x} = -dV/dx$. Sketch the potential as a function of r. Be sure to show all the qua tatively different cases, including bifurcation values of r.

a) (Saddle-node) $\dot{x} = r - x^2$
b) (Transcritical) $\dot{x} = rx - x^2$
c) (Subcritical pitchfork) $\dot{x} = rx + x^3 - x^5$

3.5 Overdamped Bead on a Rotating Hoop

3.5.1 Consider the bead on the rotating hoop discussed in Section 3.5. Expla in physical terms why the bead cannot have an equilibrium position with $\phi > \pi/$

3.5.2 Do the linear stability analysis for all the fixed points for Equation (3.5.7 and confirm that Figure 3.5.6 is correct.

3.5.3 Show that Equation (3.5.7) reduces to $\frac{d\phi}{d\tau} = A\phi - B\phi^3 + O(\phi^5)$ ne $\phi = 0$. Find A and B.

3.5.4 (Bead on a horizontal wire) A bead of mass m is constrained to slide alon a straight horizontal wire. A spring of relaxed length L_0 and spring constant k is a tached to the mass and to a support point a distance h from the wire (Figure 1).

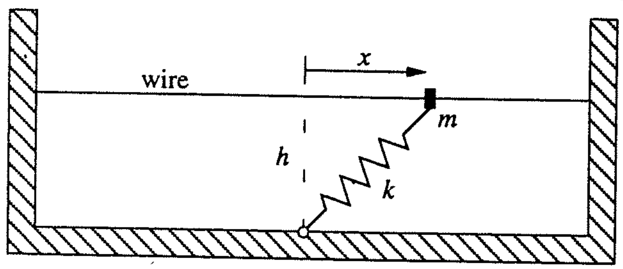

Figure 1

Finally, suppose that the motion of the bead is opposed by a viscous damping forc $b\dot{x}$.

a) Write Newton's law for the motion of the bead.
b) Find all possible equilibria, i.e., fixed points, as functions of $k, h, m, b,$ and L_0
c) Suppose $m = 0$. Classify the stability of all the fixed points, and draw a bifurca tion diagram.
d) If $m \neq 0$, how small does m have to be to be considered negligible? In wha sense is it negligible?

3.5.5 (Time scale for the rapid transient) While considering the bead on the rotating hoop, we used phase plane analysis to show that the equation

$$\varepsilon \frac{d^2\phi}{d\tau^2} + \frac{d\phi}{d\tau} = f(\phi)$$

has solutions that rapidly relax to the curve where $\frac{d\phi}{d\tau} = f(\phi)$.

a) Estimate the time scale T_{fast} for this rapid transient in terms of ε, and then express T_{fast} in terms of the original dimensional quantities m, g, r, ω, and b.
b) Rescale the original differential equation, using T_{fast} as the characteristic time scale, instead of $T_{slow} = b/mg$. Which terms in the equation are negligible on this time scale?
c) Show that $T_{fast} << T_{slow}$ if $\varepsilon << 1$. (In this sense, the time scales T_{fast} and T_{slow} are *widely separated*.)

3.5.6 (A model problem about singular limits) Consider the *linear* differential equation

$$\varepsilon\ddot{x} + \dot{x} + x = 0,$$

subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = 0$.

a) Solve the problem analytically for all $\varepsilon > 0$.
b) Now suppose $\varepsilon << 1$. Show that there are two widely separated time scales in the problem, and estimate them in terms of ε.
c) Graph the solution $x(t)$ for $\varepsilon << 1$, and indicate the two time scales on the graph.
d) What do you conclude about the validity of replacing $\varepsilon\ddot{x} + \dot{x} + x = 0$ with its singular limit $\dot{x} + x = 0$?
e) Give two physical analogs of this problem, one involving a mechanical system, and another involving an electrical circuit. In each case, find the dimensionless combination of parameters corresponding to ε, and state the physical meaning of the limit $\varepsilon << 1$.

3.5.7 (Nondimensionalizing the logistic equation) Consider the logistic equation $\dot{N} = rN(1 - N/K)$, with initial condition $N(0) = N_0$.

a) This system has three dimensional parameters r, K, and N_0. Find the dimensions of each of these parameters.
b) Show that the system can be rewritten in the dimensionless form

$$\frac{dx}{d\tau} = x(1 - x), \qquad x(0) = x_0$$

when $g = 0$; this case is the borderline between supercritical and subcritical b[ifurcations]. Find the relation between A^* and ε when $g = 0$.

c) In experiments on Taylor–Couette vortex flow, Aitta et al. (1985) were able [to] change the parameter g continuously from positive to negative by varying th[e] aspect ratio of their experimental set-up. Assuming that the equation is mod[i]fied to $\tau\dot{A} = h + \varepsilon A - gA^3 - kA^5$, where $h > 0$ is a slight imperfection, sketch th[e] bifurcation diagram of A^* vs. ε in the three cases $g > 0$, $g = 0$, and $g < 0$. Then look up the actual data in Aitta et al. (1985, Figure 2) or see Ahle[rs] (1989, Figure 15).

d) In the experiments of part (c), the amplitude $A(t)$ was found to evolve toward[s] steady state in the manner shown in Figure 2 (redrawn from Ahlers (1989), Fig[ure] 18). The results are for the imperfect subcritical case $g < 0$, $h \neq 0$. In th[e] experiments, the parameter ε was switched at $t = 0$ from a negative value to [a] positive value ε_f. In Figure 2, ε_f increases from the bottom to the top.

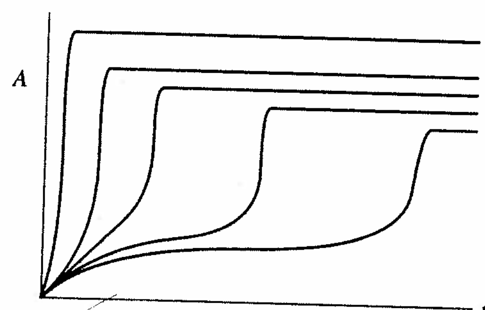

Figure 2

Explain intuitively why the curves have this strange shape. Why do the curves fo[r] large ε_f go almost straight up to their steady state, whereas the curves for small ε_f rise to a plateau before increasing sharply to their final level? (Hint: Graph $\dot{A}$ vs. A for different ε_f.)

3.6.7 (Simple model of a magnet) A magnet can be modeled as an enormous collection of electronic spins. In the simplest model, known as the ***Ising model***, the spins can point only up or down, and are assigned the values $S_i = \pm 1$, for $i = 1, \ldots, N >> 1$. For quantum mechanical reasons, the spins like to point in the same direction as their neighbors; on the other hand, the randomizing effects of temperature tend to disrupt any such alignment.

An important macroscopic property of the magnet is its average spin or *magnetization*

$$m = \left| \frac{1}{N} \sum_{i=1}^{N} S_i \right|.$$

At high temperature the spins point in random directions and so $m \approx 0$; the material is in the *paramagnetic* state. As the temperature is lowered, m remains near zero until a critical temperature T_c is reached. Then a ***phase transition*** occurs and the material spontaneously magnetizes. Now $m > 0$; we have a *ferromagnet*.

But the symmetry between up and down spins means that there are *two* possible ferromagnetic states. This symmetry can be broken by applying an external magnetic field h, which favors either the up or down direction. Then, in an approximation called *mean-field theory*, the equation governing the equilibrium value of m is

$$h = T \tanh^{-1} m - Jnm$$

where J and n are constants; $J > 0$ is the ferromagnetic coupling strength and n is the number of neighbors of each spin (Ma 1985, p. 459).

a) Analyze the solutions m^* of $h = T\tanh^{-1} m - Jnm$, using a graphical approach.

b) For the special case $h = 0$, find the critical temperature T_c at which a phase transition occurs.

3.7 Insect Outbreak

3.7.1 (Warm-up question about insect outbreak model) Show that the fixed point $x^* = 0$ is *always unstable* for Equation (3.7.3).

3.7.2 (Bifurcation curves for insect outbreak model)

a) Using Equations (3.7.8) and (3.7.9), sketch $r(x)$ and $k(x)$ vs. x. Determine the limiting behavior of $r(x)$ and $k(x)$ as $x \to 1$ and $x \to \infty$.

b) Find the exact values of r, k, and x at the cusp point shown in Figure 3.7.5.

√**3.7.3** (A model of a fishery) The equation $\dot{N} = rN(1 - \frac{N}{K}) - H$ provides an extremely simple model of a fishery. In the absence of fishing, the population is assumed to grow logistically. The effects of fishing are modeled by the term $-H$, which says that fish are caught or "harvested" at a constant rate $H > 0$, independent of their population N. (This assumes that the fishermen aren't worried about fishing the population dry—they simply catch the same number of fish every day.)

a) Show that the system can be rewritten in dimensionless form as

$$\frac{dx}{d\tau} = x(1 - x) - h,$$

for suitably defined dimensionless quantities x, τ, and h.

b) Plot the vector field for different values of h.

c) Show that a bifurcation occurs at a certain value h_c, and classify this bifurcation.

d) Discuss the long-term behavior of the fish population for $h < h_c$ and $h > h_c$, and give the biological interpretation in each case.

There's something silly about this model—the population can become nega-