75

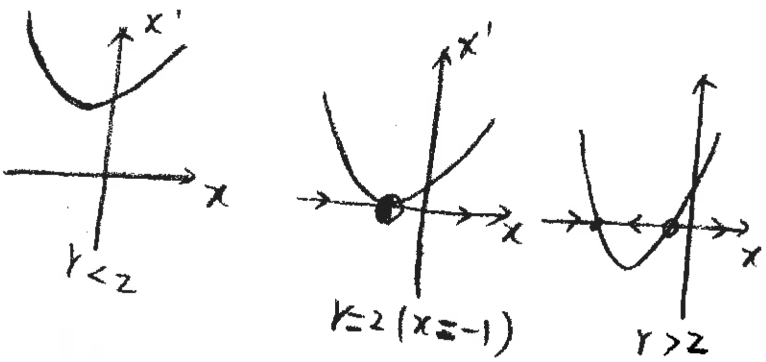
$\boxed{3.1.1}$ $\dot{x} = 1 + rx + x^2$ 5

Method 1: $f(x) = 1 + rx + x^2 \Rightarrow$

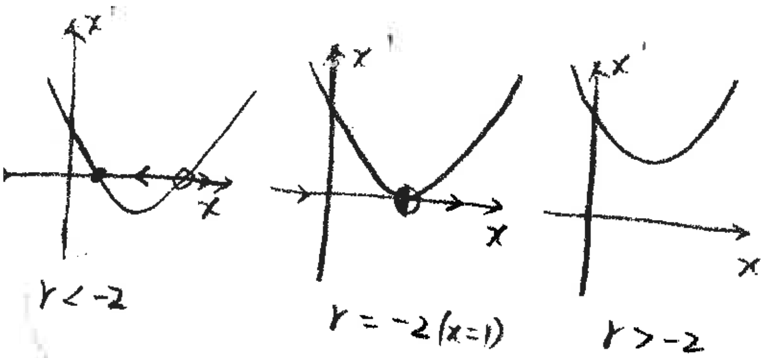
$$x_1^* = \frac{-r+\sqrt{r^2-4}}{2}, \quad x_2^* = \frac{-r-\sqrt{r^2-4}}{2}$$

The vector field:

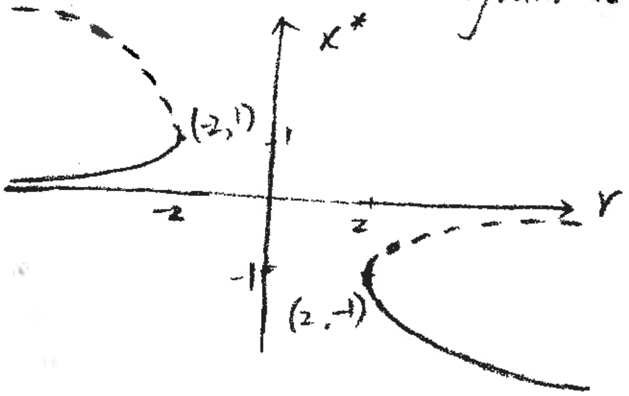
(1) $r > 0$



(2) $r < 0$



Thus a saddle-node bifurcation occurs at $r_c = \pm 2$, and the diagram is:



Method 2: $f(x) = 1 + rx + x^2$

If $r^2 < 4$ i.e. $-2 < r < 2$, there is no fixed point.

If $r^2 = 4$ i.e. $r = 2$ or -2, there is only one fixed point $x^* = -1$ when $r = 2$

$x^* = 1$ when $r = -2$

If $r^2 > 4$ i.e. $r < -2$ or $r > 2$, there are two fixed points:

$$x_1^* = \frac{-r+\sqrt{r^2-4}}{2}, \quad x_2^* = \frac{-r-\sqrt{r^2-4}}{2}$$

As for the stability, we have

$f'(x) = r + 2x$

$f'(x_1^*) = r + (-r + \sqrt{r^2-4}) = \sqrt{r^2-4} > 0$,

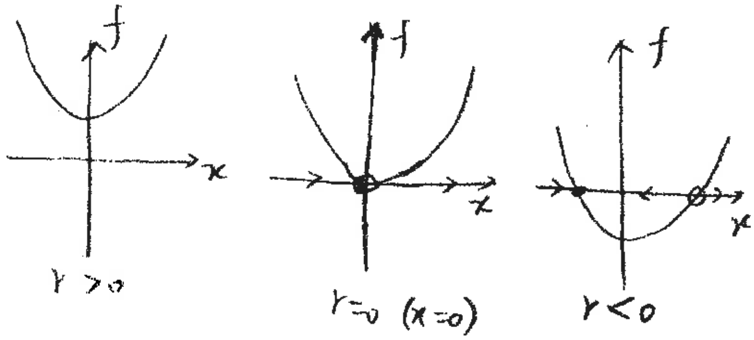
$f'(x_2^*) = r + (-r - \sqrt{r^2-4}) = -\sqrt{r^2-4} < 0$

Thus the bifurcation occurs at $r_c = \pm 2$, and when $r^2 > 4$, x_1^* is unstable, x_2^* is stable. The diagram is the same as above.

$\boxed{3.1.3.}$ $x' = r + x - \ln(1+x)$ 5

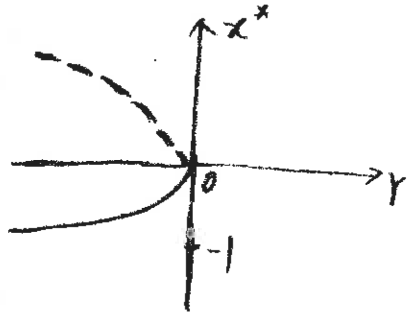
Method 1: $f(x) = r + x - \ln(1+x)$

The vector field is:



Thus the bifurcation occurs at $r_c = 0$,

and the diagram is

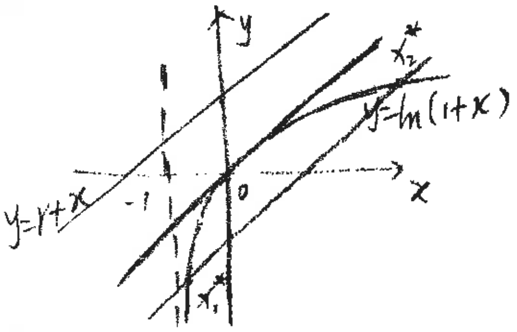


Method 2: $f(x) = r + x - \ln(1+x) = 0$

$\Rightarrow r + x = \ln(1+x)$ ①

The fixed points are the intersections

of the line $y = r + x$ and the curve $y = \ln(1+x)$



Take the derivative of ①: $1 = \frac{1}{1+x}$

$\Rightarrow x^* = 0$ From $f(x^*) = 0$, we have $r_c = 0$

$r > 0$, no fixed point

$r = 0$, one fixed point $x^* = 0$.

$r < 0$, two fixed points $-1 < x_1^* < 0$,

$x_2^* > 0$

$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}$

$f'(x_1^*) < 0 \Rightarrow x_1^*$ is stable

$f'(x_2^*) > 0 \Rightarrow x_2^*$ is unstable.

The bifurcation occurs at $r_c = 0$,

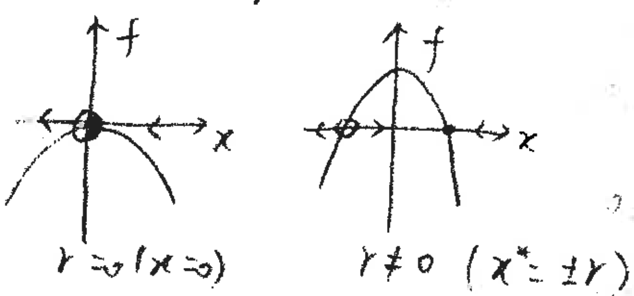
and the diagram is the same as abo

$\boxed{3.1.5}$ 5

a) $x' = r^2 - x^2$

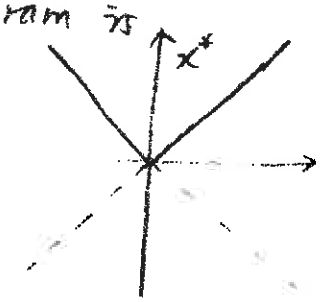
Method 1: $f(x) = r^2 - x^2$

The vector field is



The bifurcation occurs at $r_c = 0$, and

the diagram is



Method 2: $f(x)=0 \Rightarrow x_1^*=r,\ x_2^*=-r$

$f'(x)=-2x$

$f'(x_1^*)=-2r \begin{cases} >0, & r<0 \Rightarrow x_1^* \text{ unstable} \\ <0, & r>0 \Rightarrow x_1^* \text{ stable} \end{cases}$

$f'(x_2^*)=2r \begin{cases} >0, & r>0 \Rightarrow x_2^* \text{ unstable} \\ <0, & r<0 \Rightarrow x_2^* \text{ stable} \end{cases}$

The bifurcation occurs at $r_c=0$, and the diagram is the same as above.

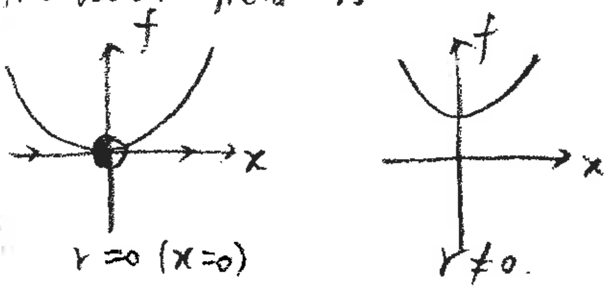
b) $x'=r^2+x^2$ 5

$f(x)=r^2+x^2=0$

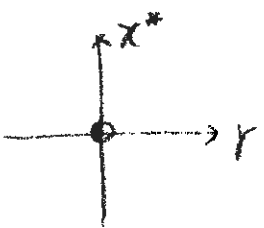
$r=0$, there is only one fixed point $x^*=0$;

$r\neq 0$, there is no fixed point.

The vector field is



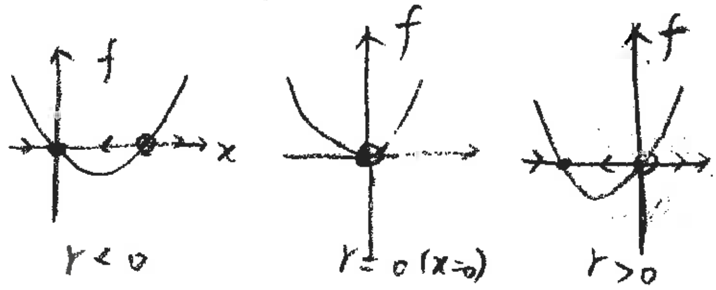
So the fixed point as a function of r is:



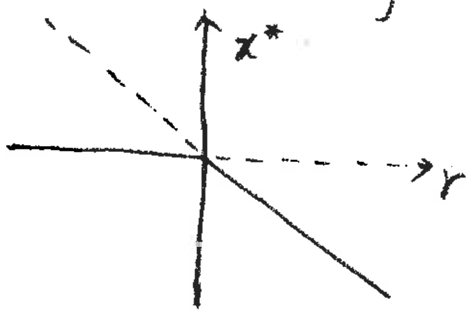
$\boxed{3.2.1}$ $x'=rx+x^2$ 5

Method 1: $f(x)=rx+x^2=0 \Rightarrow x_1^*=0,\ x_2^*=-r$

The vector field is:



Thus a transcritical bifurcation occurs at $r_c=0$, and the diagram is:



Method 2: $f'=r+2x$

$f'(x_1^*)=r \begin{cases} >0, & r>0, & x_1^* \text{ unstable} \\ <0, & r<0, & x_1^* \text{ stable} \end{cases}$

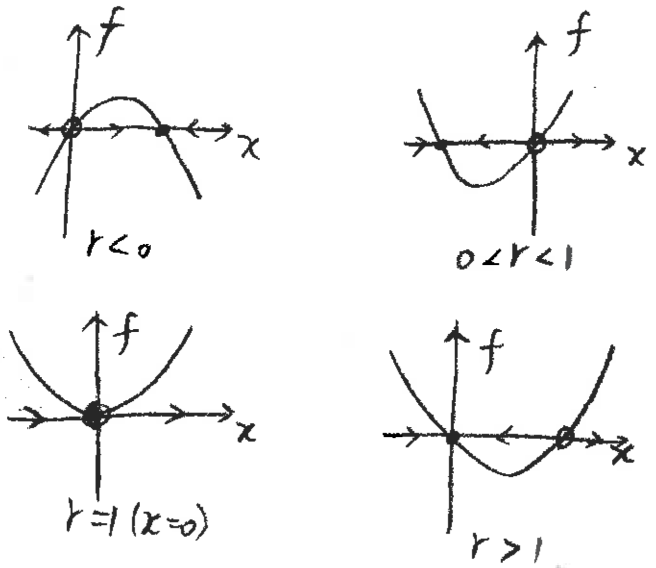
$f'(x_2^*)=-r \begin{cases} >0, & r<0, & x_2^* \text{ unstable} \\ <0, & r>0, & x_2^* \text{ stable} \end{cases}$

The bifurcation occurs at $r_c=0$

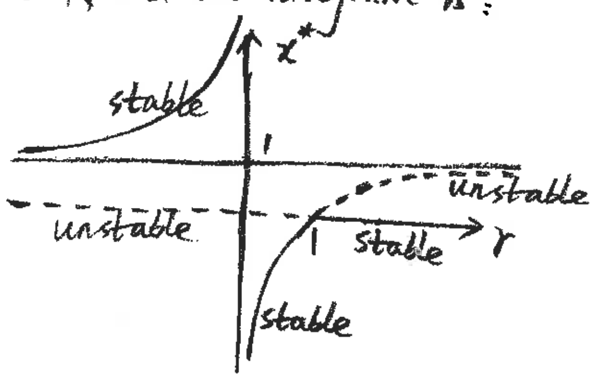
$\boxed{3.2.3}$ 5 $x'=x-rx(1-x)$

$f(x)=x-rx(1-x)=0 \Rightarrow x_1^*=0,\ x_2^*=1-\frac{1}{r}$

The vector field is



Thus a transcritical bifurcation occurs at $r_c=1$, and the diagram is:



Method 2:

$f' = 1 - r + 2rx$

$f'(x_1^*) = f'(0) = 1 - r \begin{cases} >0,\ r<1 \Rightarrow x_1^* \text{ unstable} \\ <0,\ r>1 \Rightarrow x_1^* \text{ stable} \end{cases}$

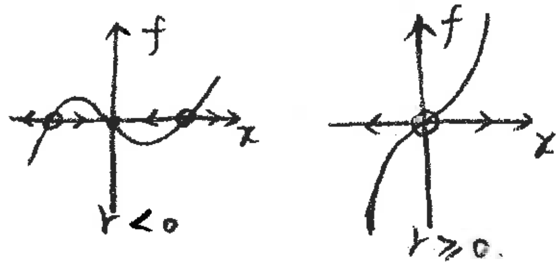
$f'(x_2^*) = f'(1-\frac{1}{r}) = -1 + r \begin{cases} >0,\ r>1 \Rightarrow x_2^* \text{ unstable} \\ <0,\ r<1 \Rightarrow x_2^* \text{ stable} \end{cases}$

The bifurcation occurs at $r_c = 1$.

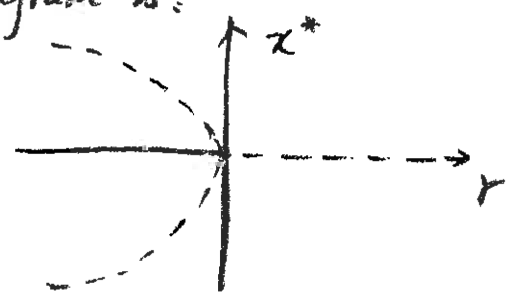
$\boxed{3.4.1}$ 5 $\dot{x} = rx + 4x^3$

$f(x) = rx + 4x^3 = 0 \Rightarrow x_1^* = 0,\ r + 4x^2 = 0$.

The vector field is:



Thus a subcritical pitchfork bifurcation occurs at $r_c = 0$, and the diagram is:



Method 2:

$f(x) = x(r + 4x^2)$, $f' = r + 12x^2$

① $r \geq 0$, one fixed point, $x_1^* = 0$

and $f'(0) = r > 0$ (when $r>0$), $x_1^* = 0$ unsta

② $r<0$, three fixed points, $x_1^* = 0$,

$x_2^* = \frac{\sqrt{-r}}{2}$, $x_3^* = \frac{-\sqrt{-r}}{2}$

$f'(x_1^*) = r < 0 \Rightarrow x_1^*$ stable

$f'(x_2^*) = f'(x_3^*) = r + 3(-r) = -2r > 0$

$\Rightarrow x_2^*, x_3^*$ unstable

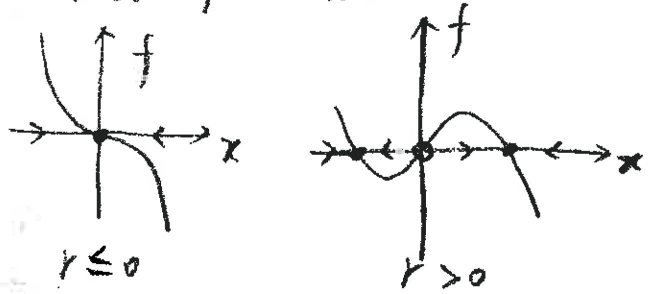
subcritical pitchfork bifurcation occurs at $r_c = 0$

4

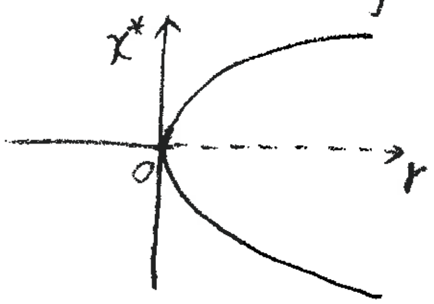
3.4.3 $x' = rx - 4x^3$ 5

$f(x) = rx - 4x^3 = 0 \Rightarrow x_1^* = 0, \quad r - 4x^2 = 0$

The vector field is:



Thus a supercritical pitchfork bifurcation occurs at $r_c = 0$, and the diagram is:



Method 2: $f' = r - 12x^2$

use the local stability, similar as 3.4.1

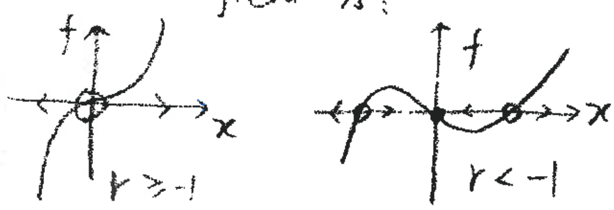
5

3.4.4 $x' = x + \frac{rx}{1+x^2}$

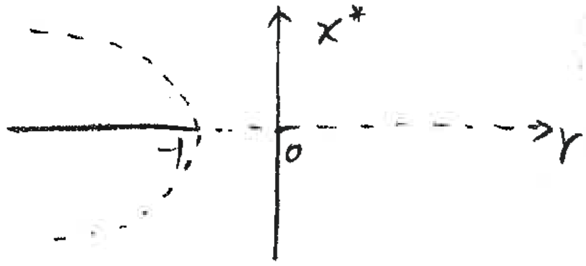
$f(x) = x + \frac{rx}{1+x^2} = x\left(1 + \frac{r}{1+x^2}\right) = 0$

$\Rightarrow x_1^* = 0$ and $x_2^* = \sqrt{-1-r}$, $x_3^* = -\sqrt{-1-r}$

The vector field is:



Thus a subcritical pitchfork bifurcation occurs at $r_c = -1$, and the diagram is:



3.4.5 $x' = r - 3x^2$ 5

$f(x) = r - 3x^2 = 0 \Rightarrow x^2 = \frac{r}{3}$

$r < 0$, no fixed point

$r = 0$, one fixed point $x = 0$

$r > 0$, two fixed points $x_1^* = \sqrt{\frac{r}{3}}$, $x_2^* = -\sqrt{\frac{r}{3}}$

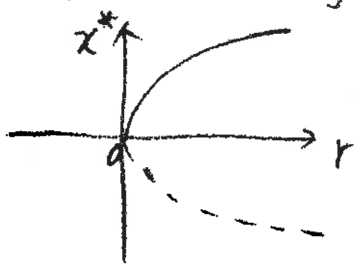
A saddle-node bifurcation occurs at $r_c = 0$ $\qquad f' = -6x$

$r > 0$:

$f'(x_1^*) < 0 \Rightarrow x_1^*$ stable

$f'(x_2^*) > 0 \Rightarrow x_2^*$ unstable

The bifurcation diagram:



3.4.7 $x' = 5 - re^{-x^2}$ 5

$f(x) = 5 - re^{-x^2} = 0$ i.e. $-x^2 = \ln\left(\frac{5}{r}\right)$

$\Rightarrow x^2 = \ln\frac{r}{5}$

$r<5$, no fixed point

$r=5$, one fixed point $x^*=0$

$r>5$, two fixed points:

$$x_1^* = \sqrt{\ln\frac{r}{5}}, \quad x_2^* = -\sqrt{\ln\frac{r}{5}}$$

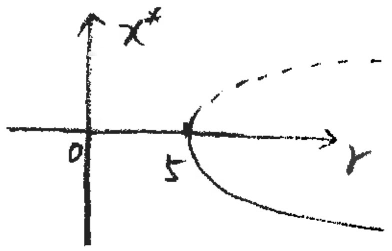
$$f' = r2xe^{-x^2}$$

$r>5$:

$f'(x_1^*)>0 \Rightarrow x_1^*$ unstable

$f'(x_2^*)<0 \Rightarrow x_2^*$ stable.

Thus a saddle-node bifurcation occurs at $r_c=5$, and the diagram is



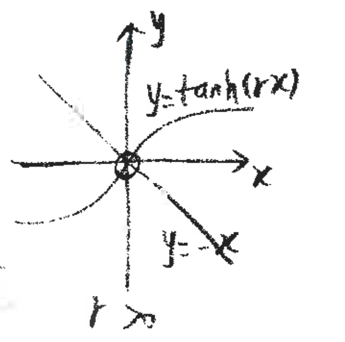
5

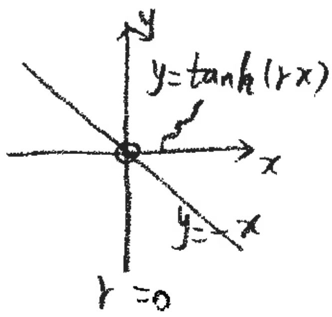
$\boxed{3.4.9}$ $x' = x + \tanh(rx)$

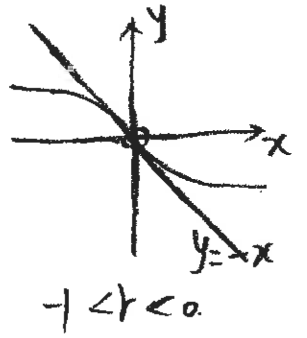
$f(x) = x + \tanh(rx) = 0$

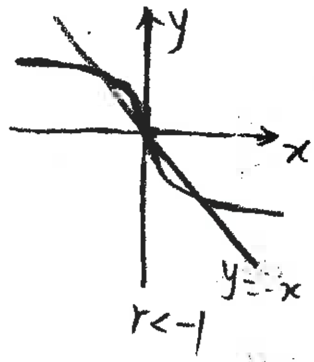
i.e. $\tanh(rx) = -x$

The fixed points are the intersections of the line $y=-x$ and the curve $y=\tanh(rx)$





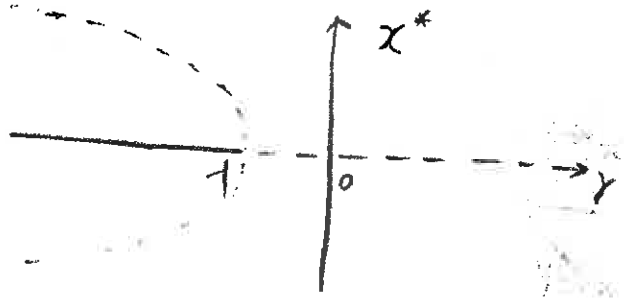




$r>-1$, only one fixed point $x^*=0$ unstable

$r<-1$, three fixed points.

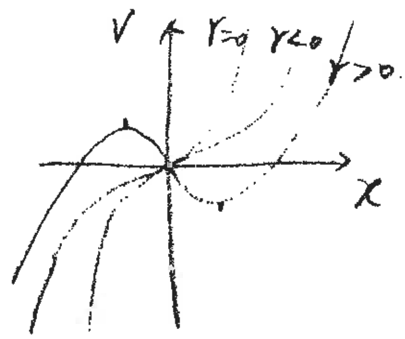
$x_1^*=0$ stable, the other two unsta



A subcritical pitchfork bifurcation occurs at $r_c=-1$.

3.4.16 5

$$\frac{dV}{dx} = x^2 - r \Rightarrow V(x) = \frac{1}{3}x^3 - rx$$



A saddle node bifurcation occurs at $r_c=0$

$r>0$: $\exists$ local minimums and local maximum

P. 6

3.7.3 (10)

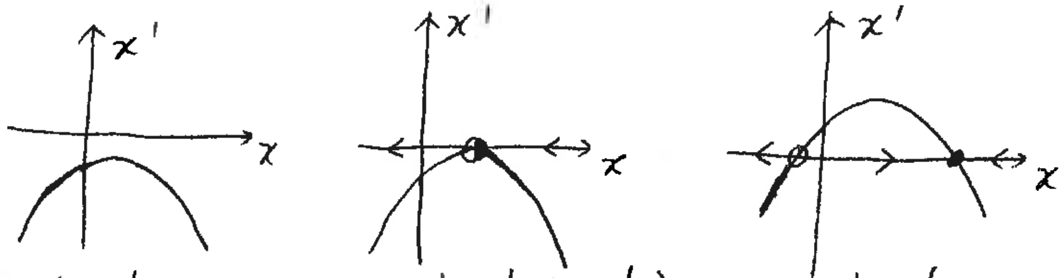
a) Let $x=\frac{N}{k}$, $\tau=rt$, $h=\frac{H}{kr}$, then

$\frac{dN}{dt}=\frac{k\,dx}{\frac{1}{r}d\tau}$, and $rN(1-\frac{N}{k})-H=rxk(1-x)-hkr$.

$\Rightarrow \frac{dN}{dt}=rN(1-\frac{N}{k})-H$ becomes: $\frac{k\,dx}{\frac{1}{r}d\tau}=rk\,x(1-x)-hkr$ i.e. $\frac{dx}{d\tau}=x(1-x)-h$

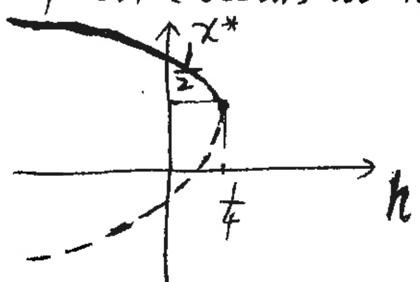
b) $f(x)=x(1-x)-h)=-x^2+x-h=0 \Rightarrow x^*=\frac{1\pm\sqrt{1-4h}}{2}$

The vector field is:



$h>\frac{1}{4}$ $\quad$ $h=\frac{1}{4}\ (x=\frac{1}{2})$ $\quad$ $h<\frac{1}{4}$

c) The saddle-node bifurcation occurs at $h_c=\frac{1}{4}$, $x^*=\frac{1}{2}$, and the diagram is:



d) If $h>h_c$, we can see $f(x)<0$ for any x, which implies the population is decreasing even to negative value.

If $h<h_c$, we can see there is a positive stable fixed point $x^*=\frac{1+\sqrt{1-4h}}{2}$.

So the population will approach this equilibrium x^*.