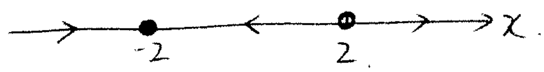
2.2.1 $\dot{x} = 4x^2 - 16$.

Solution: $f(x) = 4x^2 - 16 = 0$, then we obtain the fixed points are: $x_1^* = 2$, $x_2^* = -2$.

$x \in (-\infty, -2) \cup (2, +\infty)$, $f(x) > 0$;

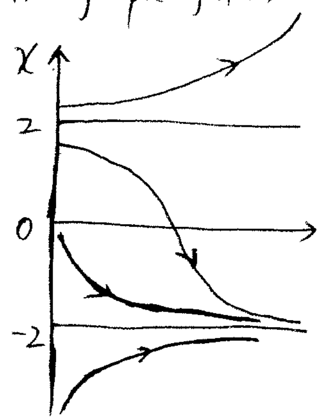
$x \in (-2, 2)$, $f(x) < 0$.

So the phase portrait is:



$x_1^* = 2$ is unstable, and $x_2^* = -2$ is stable.

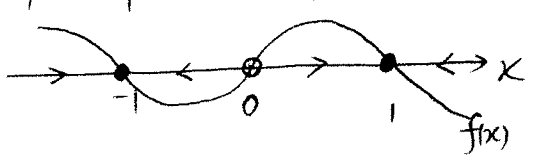
The graph of $x(t)$ is:



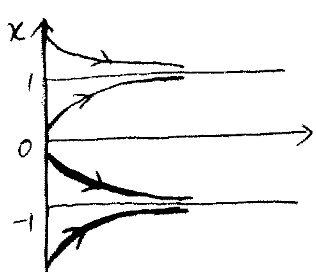
2.2.3 $x' = x - x^3$.

Solution: $f(x) = x - x^3 = 0$, then we obtain the fixed points are: $x_1^* = 0$, $x_2^* = -1$, $x_3^* = 1$.

the phase portrait is:



$x_1^* = 0$ is unstable, $x_2^* = -1$ and $x_3^* = 1$ are stable.

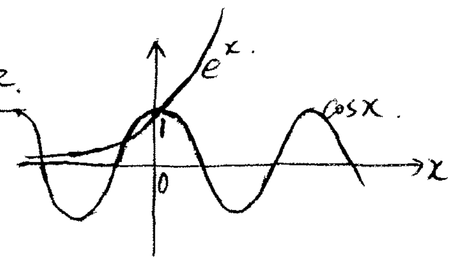


2.2.5 $x' = 1 + \frac{1}{2}\cos x$.

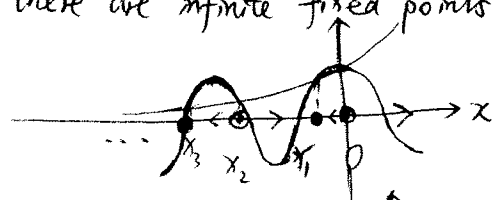
Solution: $f(x) = 1 + \frac{1}{2}\cos x = 0$, then we obtain ~~the fixed points are:~~ that there is no fixed point.

2.2.7 $x' = e^x - \cos x$.

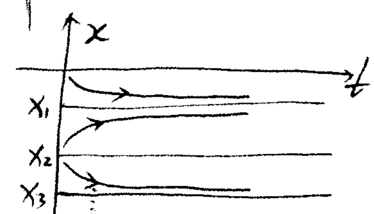
Solution: $f(x) = e^x - \cos x = 0$



there are infinite fixed points.



The graph of $x(t)$:



2.2.8:

three fixed points: $x_1^* = -1$, $x_2^* = 0$, $x_3^* = 2$.

$x_1^* = -1$ is half-stable, so $f(x)$ has the term $(x+1)^2$; and it also should have term x, $(x-2)$

So $f(x) = \cancel{x}(x+1)^2(x-2)$.

i.e. $x' = \cancel{x}(x+1)^2(x-2)$.

2.3.1

Solution: (a) $N' = rN(1 - \frac{N}{K})$.

i.e. $\frac{dN}{rN(1-\frac{N}{K})} = dt$.

$$\frac{1}{r}\int \frac{dN}{N(1-\frac{N}{K})} = \int dt.$$

$$\frac{1}{r}\int \frac{K\,dN}{N(K-N)} = \int dt.$$

$$-\frac{1}{r}\left[\int \frac{dN}{N-K} - \int \frac{dN}{N}\right] = t + c.$$

$$-\frac{1}{r}\ln\frac{N-K}{N} = t + C$$

$$\frac{N-K}{N} = Ce^{-rt}.$$

$$1-\frac{K}{N} = ce^{-rt}.$$

$\Rightarrow N = \dfrac{K}{1-ce^{-rt}}$, where C is a constant.

Since $N(0) = \dfrac{K}{1-C} = N_0$, we have $C = 1-\dfrac{K}{N_0}$

$$\text{So, } N(t) = \frac{K}{1-(1-\frac{K}{N_0})e^{-rt}} = \frac{N_0Ke^{rt}}{N_0(e^{rt}-1)+K}.$$

(b). Let $x=\frac{1}{N}$, i.e. $N=\frac{1}{x}$, then $N' = -\frac{1}{x^2}x'$.

So $-\frac{1}{x^2}x' = r\cdot\frac{1}{x}\left(1-\frac{1}{x}\cdot\frac{1}{K}\right)$.

i.e. $-x' = r\left(x-\frac{1}{K}\right)$

$\Rightarrow x' = -rx + \frac{r}{K}$.

From this linear ODE, we know $x=\frac{1}{K}$ is a special solution. Thus $x(t) = ce^{-rt}+\frac{1}{K}$.

Then $N(t) = \dfrac{1}{x(t)} = \dfrac{1}{ce^{-rt}+\frac{1}{K}} = \dfrac{K}{Kce^{-rt}+1}$.

Since $N(0) = \dfrac{K}{Kc+1} = N_0$, we have $Kc = \dfrac{K}{N_0}-1$.

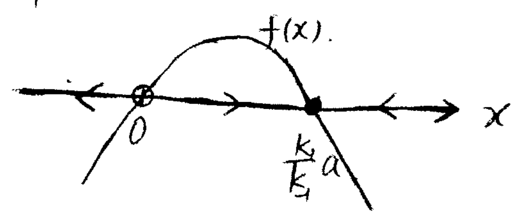
Therefore $N(t) = \dfrac{K}{(\frac{K}{N_0}-1)e^{-rt}+1} = \dfrac{N_0Ke^{rt}}{K+N_0(e^{rt}-1)}$.

2.3.2 $\quad x' = k_1ax - k_{-1}x^2$.

Solution: (a) $f(x) = k_1ax - k_{-1}x^2$, then we obtain the fixed points are:

$$x_1^* = 0,\quad x_2^* = \frac{k_1}{k_{-1}}a.$$

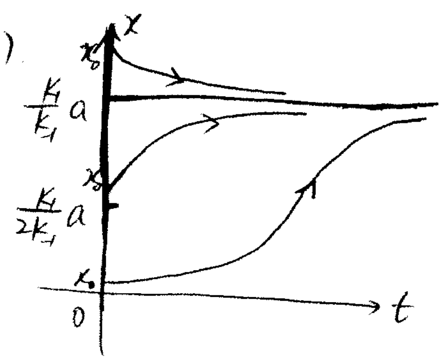
Since a, k_1, k_{-1} are all positive, we know that x_2^* is a positive fixed point. So the phase portrait is:



Hence, $x_1^*=0$ is unstable,

$x_2^* = \frac{k_1}{k_{-1}}a$ is stable.

(b)



2.3.4

Solution:

(a) $\dfrac{N'}{N} = r - a(N-b)^2$.

when $N=b$, $\left(\frac{N'}{N}\right)_{max} = r$.

and $N\in(0,b)$, $\frac{N'}{N}$ is increasing;

D

$N \in (b, +\infty)$, $\frac{N'}{N}$ is decreasing.

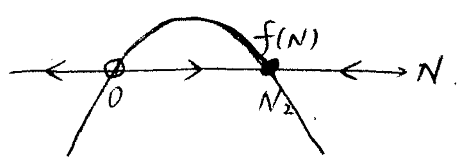
i.e. the effective growth rate $\frac{N'}{N}$ is highest at intermediate $N = b$. This is the Allee effect.

(b). $\cancel{f(x) = r - a}$ We know: $N' = rN - aN(N-b)^2$

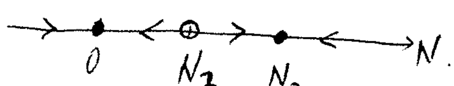
$f(N) = rN - aN(N-b)^2 = 0$, we obtain the fixed points are: $N_1^* = 0$, $N_2^* = b + \sqrt{\frac{r}{a}}$, $N_3^* = b - \sqrt{\frac{r}{a}}$.

i) If $b = \sqrt{\frac{r}{a}}$ i.e. $r = ab^2$, then $N_1 = 0$, $N_2 = b + \sqrt{\frac{r}{a}}$.

$f'(N) = r - 3aN^2 - ab^2 + 2bN$.

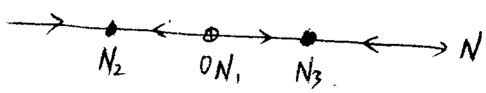


ii) If $b > \sqrt{\frac{r}{a}}$ i.e. $r < ab^2$. $f'(0) < 0$



$N_1 = 0$, $N_2 = b - \sqrt{\frac{r}{a}}$, $N_3 = b + \sqrt{\frac{r}{a}}$.

iii) If $b < \sqrt{\frac{r}{a}}$, i.e. $r > ab^2$. $f'(0) > 0$



$N_1 = 0$, $N_2 = b - \sqrt{\frac{r}{a}}$, $N_3 = b + \sqrt{\frac{r}{a}}$.

c) Turn to P_5.

2.4.1

Solution:

2.4.1 $f(x) = x(1-x)$. $\Rightarrow f'(x) = 1 - 2x$.

From $f(x) = 0$, we know the fixed points are:

$x_1^* = 0$, $x_2^* = 1$.

Since $f'(0) = 1 > 0$, and $f'(1) = -1 < 0$.

we obtain that:

$x_1^* = 0$ is unstable, $x_2^* = 1$ is stable.

2.4.3. $f(x) = \tan x$, $\Rightarrow f'(x) = \sec^2 x$.

From $f(x) = 0$, we know the fixed points are

$x^* = k\pi$, $k \in \mathbb{Z}$

Since $\sec x = 1$, $x = k\pi$ (k is even)

$= -1$, $x = k\pi$ (k is odd).

we obtain that:

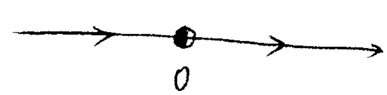
$f'(k\pi) = \sec^2(k\pi) = 1 > 0$

So, $x = k\pi$ ($k \in \mathbb{Z}$) are all unstable.

2.4.5. $f(x) = 1 - e^{-x^2} \Rightarrow f'(x) = 2xe^{-x^2}$.

From $f(x) = 0$, we know the fixed point ~~is~~ $x^* = 0$. and $f'(0) = 0$. $x \neq 0$, $f(x) > 0$.

The phase plot is:



So $x^* = 0$ is half-stable.

2.4.7 $f(x) = ax - x^3 \Rightarrow f'(x) = a - 3x^2$.

From $f(x) = 0$, we know $x = 0$ or $x^2 = a$.

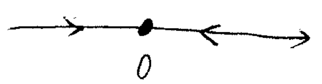
1) If a is negative, there is only one fixed point: $x^* = 0$. and $f'(x) = a - 3x^2$, $f'(0) < 0$.

~~the phase plot is~~

So $x^* = 0$ is stable.

2) If $a = 0$, then $x' = -x^3$. and there is one fixed point $x^* = 0$.

the phase plot is :



So $x^* = 0$ is stable.

3) If a is positive, we have 3 fixed points:

$x_1^* = 0$, $x_2^* = \sqrt{a}$, $x_3^* = \sqrt{a}$.

Since $f'(0) = a > 0$, $f'(\sqrt{a}) = -2a < 0$, $f'(-\sqrt{a}) < 0$.

we obtain that:

$x_1^* = 0$ is unstable, $x_2^* = \sqrt{a}$ and $x_3^* = -\sqrt{a}$ is stable.

2.7 Solution:

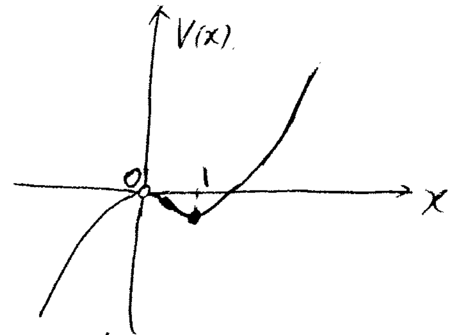
2.7.1 $x' = x(1 - x)$

We know $-\frac{dV}{dx} = x(1-x)$

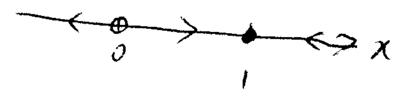
i.e. $\frac{dV}{dx} = x^2 - x$.

$\Rightarrow V(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 + C$.

$V(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 \quad (C = 0)$



equilibrium points: $x_1^* = 0$, $x_2^* = 1$



$x_1^* = 0$ is unstable.

$x_2^* = 1$ is stable.

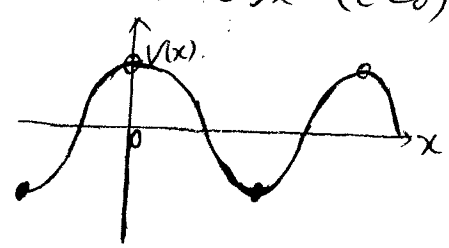
2.7.3. $x' = \sin x$.

We know $-\frac{dV}{dx} = \sin x$

i.e. $\frac{dV}{dx} = -\sin x$.

$V(x) = \cos x + C$.

$\therefore V(x) = \cos x \quad (C = 0)$



$x^* = k\pi \ (k \in \mathbb{Z})$ are equilibrium points.

and $x^* = k\pi$ (k is even) are unstable;

$x^* = k\pi$ (k is odd) are stable.

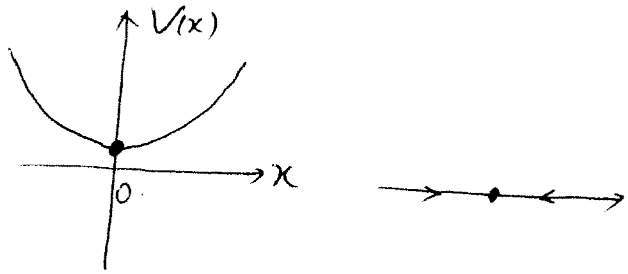
2.7.5. $x' = -\sinh x$.

we know: $-\frac{dV}{dx} = -\frac{e^x - e^{-x}}{2}$

i.e. $\frac{dV}{dx} = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$

$V(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} + C$.

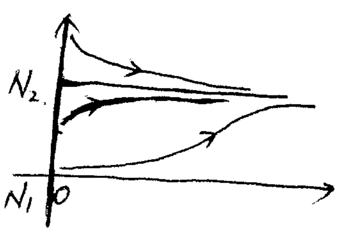
$\therefore V(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} = \cosh x \ (C=0)$



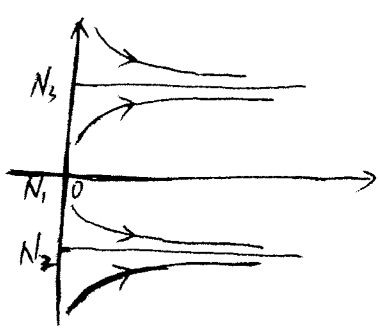
The equilibrium is $x^*=0$, and it is stable.

2.3.4.

c). If $b = \sqrt{\frac{r}{a}}$:



If $b < \sqrt{\frac{r}{a}}$:



If $b > \sqrt{\frac{r}{a}}$.