4'

$\boxed{6.3.1}$ $\dot{x} = x - y$, $\dot{y} = x^2 - 4$

Solution. From $\begin{cases} \dot{x} = x - y = 0 \\ \dot{y} = x^2 - 4 = 0 \end{cases}$, we obtain the

fixed points are: $(2, 2)$ and $(-2, -2)$.

The Jacobian matrix for the system is:

$A = \begin{pmatrix} 1 & -1 \\ 2x & 0 \end{pmatrix}$. then we have:

- $A\big|_{(2,2)} = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$, so $|A - \lambda I| = \lambda^2 - \lambda + 4 = 0$

  $\Rightarrow \lambda = \dfrac{1 \pm \sqrt{15}\, i}{2}$. $Re(\lambda) = \frac{1}{2} > 0$.

  so $(2,2)$ is an unstable spiral.

- $A\big|_{(-2,-2)} = \begin{pmatrix} 1 & -1 \\ -4 & 0 \end{pmatrix}$, so $|A - \lambda I| = \lambda^2 - \lambda - 4 = 0$

  $\Rightarrow \lambda = \dfrac{1 \pm \sqrt{17}}{2}$. $(-2,-2)$ is a saddle.

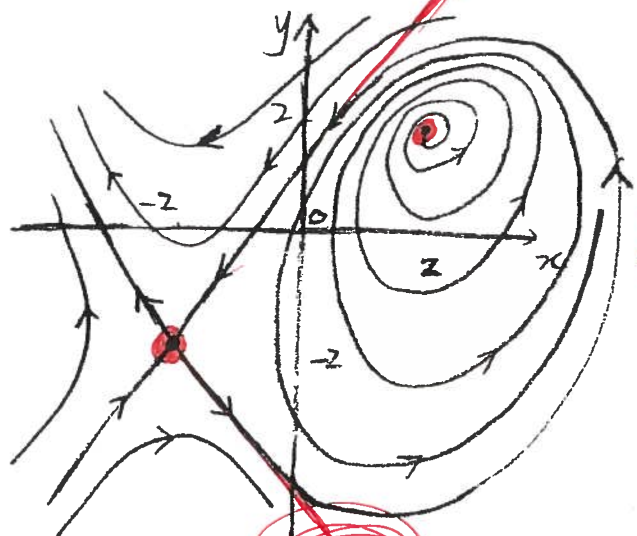
$\lambda_1 = \frac{1+\sqrt{17}}{2} > 0$: unstable

$$\begin{pmatrix} \frac{1-\sqrt{17}}{2} & -1 \\ -4 & \frac{-1-\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -0.64 \\ 1 \end{pmatrix}$$

$\lambda_2 = \frac{1-\sqrt{17}}{2} < 0$: stable.

$$\begin{pmatrix} \frac{1+\sqrt{17}}{2} & -1 \\ -4 & \frac{-1+\sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0.39 \\ 1 \end{pmatrix}$$

The phase portrait is:

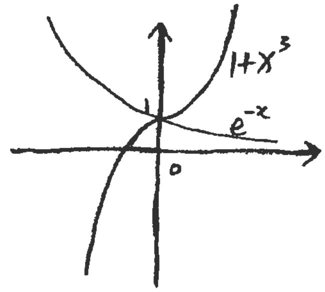


4' $\boxed{6.3.3}$ $\dot{x} = 1 + y - e^{-x}$, $\dot{y} = x^3 - y$.

Solution: From $\begin{cases} \dot{x} = 1 + y - e^{-x} = 0 \\ \dot{y} = x^3 - y = 0 \end{cases}$, we

have: $1 + x^3 - e^{-x} = 0$ i.e. $1 + x^3 = e^{-x}$.

From



we know $(0,0)$ is the only fixed point.

The Jacobian matrix is:

$A = \begin{pmatrix} e^{-x} & 1 \\ 3x^2 & -1 \end{pmatrix}$, and $A\big|_{(0,0)} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

So $|A-\lambda I| = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

$(0,0)$ is a saddle

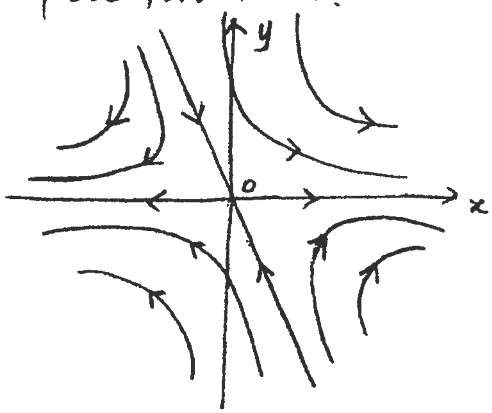
$\lambda_1 = 1$: unstable, and

$$\begin{pmatrix}0 & 1\\ 0 & -2\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

$\lambda_2 = -1$: stable, and

$$\begin{pmatrix}2 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1\\ -2\end{pmatrix}.$$

The phase portrait is:



4' $\boxed{6.3.5}$ $\dot{x} = \sin y$, $\dot{y} = \cos x$.

Solution: From $\begin{cases}\dot{x} = \sin y = 0\\ \dot{y} = \cos x = 0\end{cases}$, we obtain the fixed points are $\left(\frac{2k+1}{2}\pi, n\pi\right)$ $(k, n \in Z)$ $= (x^*, y^*)$.

The Jacobian matrix is:

$$A = \begin{pmatrix}0 & \cos y\\ -\sin x & 0\end{pmatrix},$$ then we have:

$$A\Big|_{(x^*,y^*)} = \begin{pmatrix}0 & \cos n\pi\\ -\sin\left(\frac{2k+1}{2}\pi\right) & 0\end{pmatrix}$$

$$|A-\lambda I| = \lambda^2 + \cos(n\pi)\sin\left(\frac{2k+1}{2}\pi\right) = 0$$

$$\Rightarrow \lambda_{1,2} = \pm\sqrt{-\cos(n\pi)\sin\left(\frac{2k+1}{2}\pi\right)}.$$

① If k, n are both even, then $\lambda = \pm i$, and (x^*, y^*) is a center.

② If k, n are both odd, then $\lambda = \pm i$ and (x^*, y^*) is a center.

③ If k is even, n is odd, then $\lambda = \pm 1$, (x^*, y^*) is a saddle.

$\lambda_1 = 1$: unstable, and

$$\begin{pmatrix}-1 & -1\\ -1 & -1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

$\lambda_2 = -1$: stable, and

$$\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}.$$

④ If k is odd, n is even, then $\lambda = \pm 1$, (x^*, y^*) is a saddle

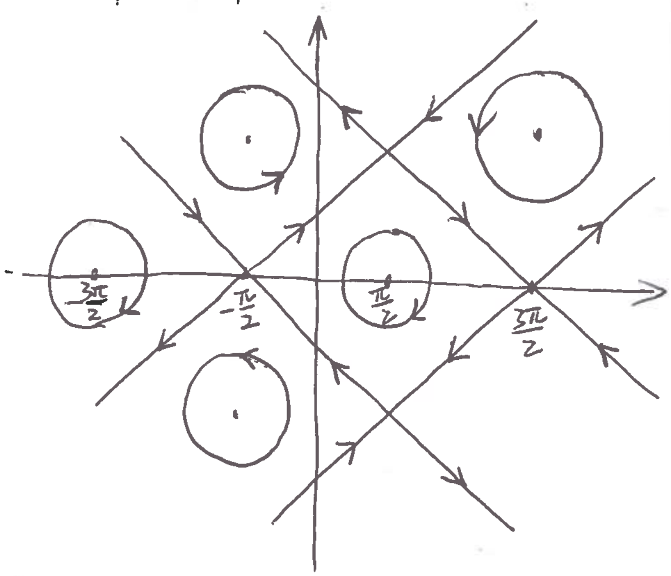
$\lambda_1 = 1$: unstable, and

$$\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}1\\ 1\end{pmatrix}$$

$\lambda_2 = -1$: stable and

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The phase portrait is:



6' $\boxed{6.3.14}$

$$\begin{cases} \dot{x} = -y + ax^3 \\ \dot{y} = x + ay^3 \end{cases}$$, $(0,0)$ is a fixed point.

Switch to the polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$, then

$x^2 + y^2 = r^2 \Rightarrow rr' = xx' + yy'$

i.e. $rr' = x(-y + ax^3) + y(x + ay^3)$

$= a(x^4 + y^4)$

$= ar^4(\cos^4\theta + \sin^4\theta)$

$\Rightarrow r' = ar^3(\cos^4\theta + \sin^4\theta)$.

$$\theta' = \frac{xy' - x'y}{r^2} = \frac{x(x + ay^3) - (-y + ax^3)y}{r^2}$$

$$= \frac{x^2 + y^2 + a(xy^3 - x^3y)}{r^2}$$

$$= 1 + a\frac{xy(y^2 - x^2)}{r^2}$$

$$= 1 + a\frac{r\cos\theta \cdot r\sin\theta(r^2\sin^2\theta - r^2\cos^2\theta)}{r^2}$$

$$= 1 + ar^2\cos\theta\sin\theta(\sin^2\theta - \cos^2\theta)$$

$$= 1 - \frac{ar^2}{4}\sin 4\theta.$$

So $\begin{cases} r' = ar^3(\cos^4\theta + \sin^4\theta) \\ \theta' = 1 - \frac{ar^2}{4}\sin 4\theta. \end{cases}$

• If $a < 0$, then $r' < 0$, and

$r \to 0$ as $t \to \infty$.

$\theta' \to 1$ as $t \to \infty$

$(0,0)$ is a stable spiral.

• If $a = 0$, then $r' = 0$, $\theta' = 1$. so

$(0,0)$ is a center

• If $a > 0$, then $r' > 0$, $r \to \infty$.

$(0,0)$ is an unstable spiral.

Hence:

$a < 0$: a stable spiral

$a = 0$: a center

$a > 0$: an unstable spiral.

$\boxed{6.3.17}$

5'

Solution. $\begin{cases} \dot{x} = xy - x^2y + y^3 \\ \dot{y} = y^2 + x^3 - xy^2 \end{cases}$ (0,0) is a fixed point.

By polar coordinates, we obtain:

$$r' = \frac{xx' + yy'}{r} = \frac{x(xy - x^2y + y^3) + y(y^2 + x^3 - xy^2)}{r}$$

$$= \frac{x^2y + y^3}{r} = \frac{r^3\cos^2\theta\sin\theta + r^3\sin^3\theta}{r}$$

$$r' = r^2\sin\theta.$$

$$\theta' = \frac{xy' - x'y}{r^2} = \frac{x(y^2 + x^3 - xy^2) - (xy - x^2y + y^3)y}{r^2}$$

$$= \frac{x^4 - y^4}{r^2} = \frac{r^4(\cos^4\theta - \sin^4\theta)}{r^2}$$

$$\theta' = r^2(\cos^4\theta - \sin^4\theta) = r^2\cos 2\theta$$

So $\begin{cases} r' = r^2\sin\theta \\ \theta' = r^2\cos 2\theta \end{cases}$

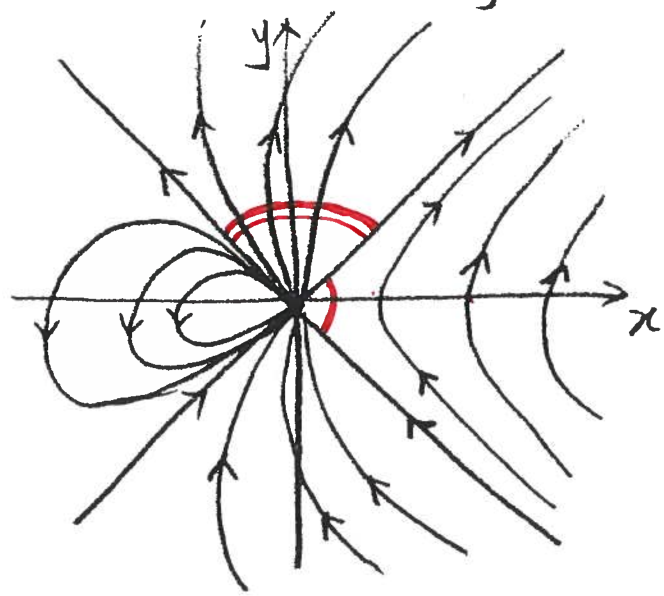
For the phase portrait, we have:

- If $y > 0$, then $r' > 0$, if $y < 0$, $r' < 0$.
- The lines $\theta = \frac{\pi}{4}, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$ are reversible. They are unstable for $y > 0$, stable for $y < 0$.
- $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$: r will reach a minimum at $\theta = 0$, since $\sin 0 = 0$

$\frac{\pi}{4} < \theta < \frac{3}{4}\pi$: θ is decreasing and r is increasing.

$\frac{3}{4}\pi < \theta < \frac{5}{4}\pi$: r will reach a maximum at $\theta = \pi$.

$\frac{5}{4}\pi < \theta < \frac{7}{4}\pi$: r is decreasing and θ is decreasing.



4' $\boxed{6.4.1}$ $\begin{cases} \dot{x} = x(3-x-y) \\ \dot{y} = y(2-x-y) \end{cases}$

From $\begin{cases} \dot{x} = 0 \\ \dot{y} = 0 \end{cases}$, we know that $\begin{cases} x=0 \\ y=0 \end{cases}$ and $\begin{cases} x+y=3 \\ x+y=2 \end{cases}$

So the fixed points are $(0,0)$, $(0,2)$, $(3,0)$

The Jacobian matrix is:

$$A = \begin{pmatrix} 3-2x-y & -x \\ -y & 2-x-2y \end{pmatrix}, \text{ then}$$

- $A|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, and $|A-\lambda I| = (3-\lambda)(2-\lambda) = 0$

  $\Rightarrow \lambda_1 = 3.\ \lambda_2 = 2$ So $(0,0)$ is an unstable node.

- $A|_{(0,2)} = \begin{pmatrix} 1 & 0 \\ -2 & -2 \end{pmatrix}$, and $|A-\lambda I| = (1-\lambda)(-2-\lambda) = 0$

  $\Rightarrow \lambda_1 = 1,\ \lambda_2 = -2$ $(0,2)$ is a saddle.

  $\lambda_1 = 1$: unstable, and

  $$\begin{pmatrix} 0 & 0 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0, \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{2}{3} \end{pmatrix}$$

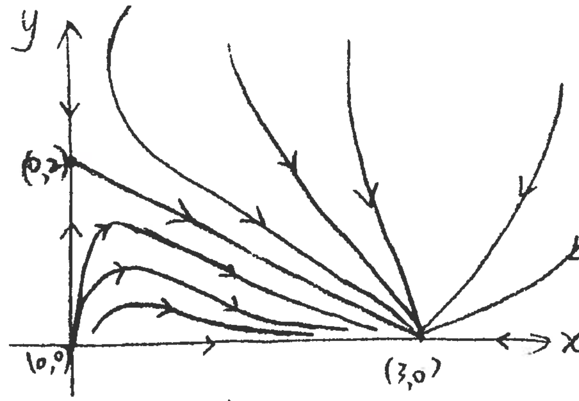
  $\lambda_2 = -2$: stable, and

  $$\begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $A|_{(3,0)} = \begin{pmatrix} -3 & -3 \\ 0 & -1 \end{pmatrix}$, and $|A-\lambda I| = (-3-\lambda)(-1-\lambda) = 0$.

  $\Rightarrow \lambda_1 = -1,\ \lambda_2 = -3$ $(3,0)$ is a stable node

The phase portrait is:



Basin of attraction at $(3,0)$.

4' $\boxed{6.4.3}$

From $\begin{cases} \dot{x} = x(3-2x-2y) = 0 \\ \dot{y} = y(2-x-y) = 0 \end{cases}$, we have

$\begin{cases} x=0 \\ y=0 \end{cases}$ and $\begin{cases} 2x+2y=3 \\ x+y=2 \end{cases}$, then

the fixed points are: $(0,0)$, $(0,2)$, $(\frac{3}{2},0)$

The Jacobian matrix is:

$$A = \begin{pmatrix} 3-4x-2y & -2x \\ -y & 2-x-2y \end{pmatrix}, \text{ then}$$

- $A|_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda_1 = 3,\ \lambda_2 = 2$.

  $(0,0)$ is an unstable node.

- $A|_{(0,2)} = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$. $\lambda_1 = -1,\ \lambda_2 = -2$

  $(0,2)$ is a stable node.

- $A|_{(\frac{3}{2},0)} = \begin{pmatrix} -3 & -3 \\ 0 & \frac{1}{2} \end{pmatrix}$, $\lambda_1 = -3,\ \lambda_2 = \frac{1}{2}$.

  $(\frac{3}{2},0)$ is a saddle.

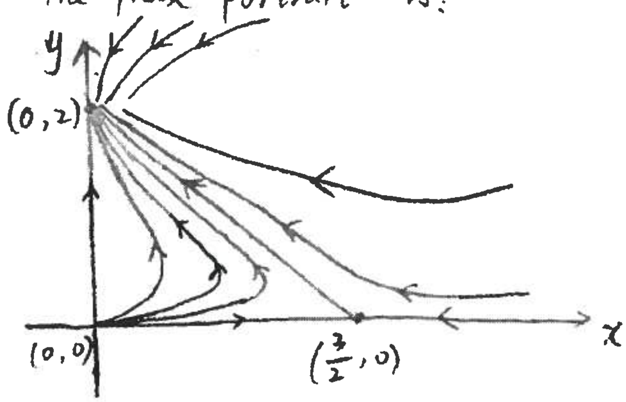
$\lambda_1 = -3$: stable, and

$$\begin{pmatrix} 0 & -3 \\ 0 & \frac{7}{2} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\lambda_2 = \frac{1}{2}$: unstable, and

$$\begin{pmatrix} -\frac{7}{2} & -3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{7}{6} \end{pmatrix}$$

The phase portrait is:



Basin of attraction at (0,2).

+1 $\boxed{6.5.2}$ $\ddot{x} = x - x^2 = F \quad (m=1)$

$$\ddot{x} = x - x^2 \Rightarrow \begin{cases} \dot{x} = y \\ \dot{y} = x - x^2 \end{cases}$$

$$V(x) = -\int F(x)\,dx = -\int (x - x^2)\,dx$$

$$= -\frac{1}{2}x^2 + \frac{1}{3}x^3$$

$$E = \frac{1}{2}\dot{x}^2 + V(x) = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{x^3}{3}$$

(a) $\begin{cases} \dot{x} = y = 0 \\ \dot{y} = x - x^2 = 0 \end{cases} \Rightarrow$ fixed points are: $(0,0)$, $(1,0)$

The Jacobian matrix is:

$$A = \begin{pmatrix} 0 & 1 \\ 1-2x & 0 \end{pmatrix}, \text{ then}$$

- $A\big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $|A - \lambda I| = \lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$. (0,0) is a saddle

$\lambda_1 = 1$: unstable, and

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

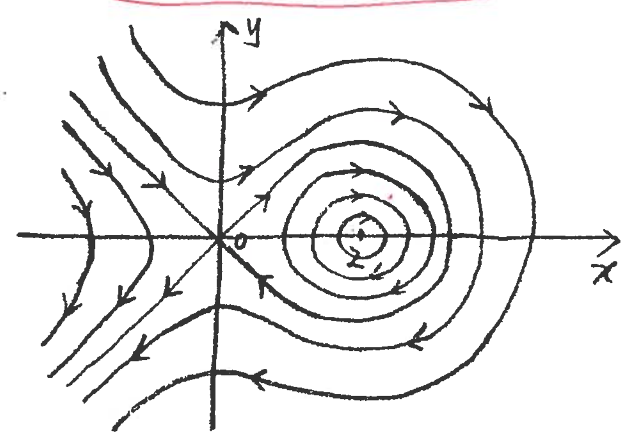
$\lambda_2 = -1$: stable, and

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

- $A\big|_{(1,0)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $|A - \lambda I| = \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$ (1,0) is a center.

(b)



(c) $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{x^3}{3} = 0$

$$\Rightarrow y^2 = x^2\left(1 - \frac{2}{3}x\right)$$

6' $\boxed{6.5.4}$ $\ddot{x} = ax - x^2$

$$\Rightarrow \begin{cases} \dot{x} = y = 0 \\ \dot{y} = ax - x^2 = 0 \end{cases}, \text{ then}$$

the fixed points are: $(0,0), (a,0)$

The Jacobian matrix is:

$$A = \begin{pmatrix} 0 & 1 \\ a-2x & 0 \end{pmatrix}, \text{ then}$$

$$A\Big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \quad A\Big|_{(a,0)} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}.$$

• $\boxed{\text{If } a>0,}$ then

① for $(0,0)$, $|A-\lambda I| = \lambda^2 - a = 0 \Rightarrow \lambda = \pm\sqrt{a}$.

$(0,0)$ is a saddle.

$\lambda_1 = \sqrt{a}$: unstable, and

$$\begin{pmatrix} -\sqrt{a} & 1 \\ a & -\sqrt{a} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{a} \end{pmatrix}$$

$\lambda_2 = -\sqrt{a}$: stable, and

$$\begin{pmatrix} \sqrt{a} & 1 \\ a & \sqrt{a} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{a} \end{pmatrix}.$$

② for $(a,0)$, $|A-\lambda I| = \lambda^2 + a = 0 \Rightarrow \lambda = \pm\sqrt{-a}$

$(a,0)$ is a center.

• $\boxed{\text{If } a<0,}$ then

$(0,0)$ is a center, and $(a,0)$ is a saddle.

$(a,0)$: $\lambda = \pm\sqrt{-a}$

$\lambda_1 = \sqrt{-a}$: unstable, and

$$\begin{pmatrix} -\sqrt{-a} & 1 \\ -a & -\sqrt{-a} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{-a} \end{pmatrix}$$

$\lambda_2 = -\sqrt{-a}$: stable, and

$$\begin{pmatrix} \sqrt{-a} & 1 \\ -a & \sqrt{-a} \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{-a} \end{pmatrix}.$$

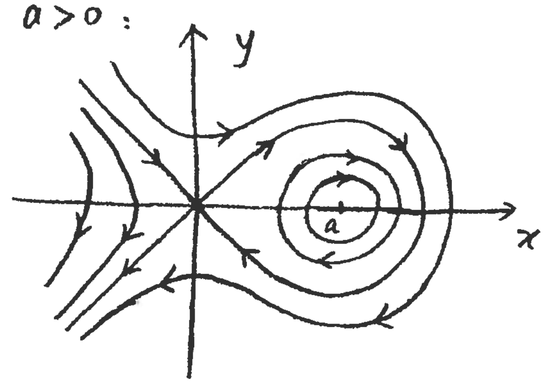
• $\boxed{\text{If } a=0,}$ then the only fixed point $(0,0)$.

$$A = \begin{pmatrix} 0 & 1 \\ -2x & 0 \end{pmatrix}, \text{ and } A\Big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

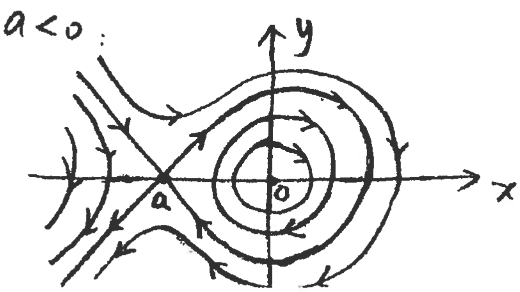
This corresponds to a degenerate node.

The phase portrait is:

(1) $a>0$:



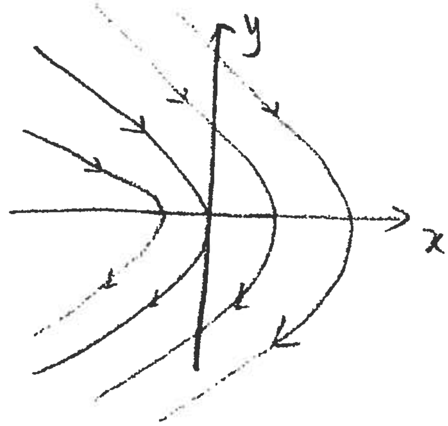
(2) $a<0$:



(3) $a=0$

$$\begin{cases} \dot{x}=y \\ \dot{y}=-x^2 \end{cases}$$

$E=\frac{1}{2}y^2+\frac{x^3}{3}=C$



$\boxed{6.5.9}$

$$H(x,p)=\frac{p^2}{2m}+\frac{kx^2}{2}$$

By chain rule, we have:

$$\frac{dH}{dt}=\frac{\partial H}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial H}{\partial p}\cdot\frac{dp}{dt}$$

Using Hamilton's equation, we obtain:

$$\frac{dx}{dt}=-\frac{\partial H}{\partial p}, \quad \frac{dp}{dt}=\frac{\partial H}{\partial x}$$

So $\frac{dH}{dt}=\frac{\partial H}{\partial x}\cdot\left(-\frac{\partial H}{\partial p}\right)+\frac{\partial H}{\partial p}\left(\frac{\partial H}{\partial x}\right)=0$.

i.e. $\dot{H}=0$

$H(x,p)$ is a constant, and is conserved.