Last Name: First name: **Student ID:**

1.

[12] 2. Find the general/particular solution for each of the following differential equations.

i) $y'' + 2y' + y = 2\,e^{-t}$

C.E. $r^2 + 2r + 1 = 0.$

$r_1 = r_2 = -1.$

$y_c = c_1 e^{-t} + c_2 t e^{-t}$

$Y(t) = At^2 e^{-t}$

$Y' = 2Ate^{-t} + (-At^2e^{-t})$

$Y'' = 2Ae^{-t} - 4Ate^{-t} + At^2e^{-t}$

sub.

$2Ae^{-t} = 2e^{-t}$

$A = 1.$

$Y = t^2e^{-t}$

$Y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$

ii) $y''' - 3y'' + 3y' - y = 0$

C.E. $r^3 - 3r^2 + 3r - 1 = 0.$

$(r-1)(r-1)(r-1) = 0$

$r_1 = r_2 = r_3 = 1$

$y = c_1e^t + c_2te^t + c_3t^2e^t$

[7] 3. Consider the initial problem $y'' + 2y' + 6y = 0,\ \ y(0) = 2,\ y'(0) = \alpha.$

a) Solve the initial value problem. (b) Determine α so that $y(1) = 0.$

a). C.E. $r^2 + 2r + 6 = 0$

$r_1 = -1 + \sqrt{5}i,\ r_2 = -1 - \sqrt{5}i$

$y(t) = c_1e^{-t}\cos\sqrt{5}t + c_2e^{-t}\sin\sqrt{5}t.$

$y(0) = 2 \Rightarrow c_1 = 2.$

$y'(t) = -c_1e^{-t}\cos\sqrt{5}t + c_1e^{-t}(-\sin\sqrt{5}t)\sqrt{5} + (-c_2e^{-t}\sin\sqrt{5}t + c_2e^{-t}\cos\sqrt{5}t\sqrt{5}).$

$y'(0) = \alpha:\ \alpha = -2 + \sqrt{5}c_2 \qquad c_2 = \frac{\alpha+2}{\sqrt{5}}$

$y = 2e^{-t}\cos\sqrt{5}t + \frac{\alpha+2}{\sqrt{5}}e^{-t}\sin\sqrt{5}t$

$y(1) = 0:\ 2\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}}\sin\sqrt{5} = 0$

$\alpha + 2 = -2\sqrt{5}\cot\sqrt{5}$

$\alpha = -2\sqrt{5}\cot\sqrt{5} - 2$

space for question 2:

[14] 4. i) If $y_1(x) = x^{-1}$ is a solution to the following equation

$$x^2y'' + 4xy' + 2y = 0.$$

find the general solution of the given equation.

ii) Based on the result in i), solve the following non-homogeneous equation

$$x^2y'' + 4xy' + 2y = 3x^2 + 1$$

1). $y_2 = x^{-1} \cdot v$. $\quad y_2' = -x^{-2} \cdot v + x^{-1} \cdot v'$

$y_2'' = 2x^{-3}v - x^{-2}v' - x^{-2}v' + x^{-1}v'' = 2x^{-3} \cdot -2x^{-2}v' + x^{-1}v''$

Sub: $\quad x^2\left(2x^{-3}v - 2x^{-2}v' + x^{-1}v''\right) + 4x\left(-x^{-2}v + x^{-1}v'\right) + 2 \cdot x^{-1}v = 0$

$x v'' + 2v' = 0. \qquad v' = \frac{c_1}{x^2}.$

$v = \int \frac{c_1}{x^2}\,dx = -\frac{c_1}{x} + c_2. \qquad y_2 = \frac{1}{x}\left(-\frac{c_1}{x} + c_2\right).$

chouse. $c_1 = -1$, $c_2 = 0$. $\quad y_2 = \frac{1}{x^2}.$

2). $g(t) = 3 + \frac{1}{x^2}$. $\quad W = \begin{vmatrix} \frac{1}{x} & \frac{1}{x^2} \\ -\frac{1}{x^2} & -2\frac{1}{x^3} \end{vmatrix} = -\frac{1}{x^4}.$

$y = y_1 \int \frac{-y_2 g}{W} + y_2 \int \frac{y_1 g}{W} = x^{-1}\int \frac{-\frac{1}{x^2}(3+\frac{1}{x^2})}{-\frac{1}{x^4}} + x^{-2}\int \frac{\frac{1}{x}(3+\frac{1}{x^2})}{-\frac{1}{x^4}}\,dx$

$= x^{-1}\int \frac{3x^2+1}{1} + x^{-2}\int \frac{x(3x^2+1)}{-1}\,dx$

$= x^{-1} \cdot \left(x^3 + x + c_1\right) + x^{-2}\left(-\frac{3}{4}x^4 - \frac{x^2}{2} + c_2\right)$

$= x^2 + 1 + c_1x^{-1} - \frac{3}{4}x^2 - \frac{1}{2} + c_2x^{-2}$

$= \frac{1}{4}x^2 + \frac{1}{2} + c_1x^{-1} + c_2x^{-2}.$

[5] 5. Show that functions $f_1 = t$, $f_2 = \cos t$, $f_3 = \sin t$ are linearly independent.

$$W = \begin{vmatrix} t & \cos t & \sin t \\ 1 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix} = t \neq 0.$$

they are lineely independent.

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[12] 6. i) Without solving the equation, choose an appropriate "form" for a particular solution of

$$y'''' + 2y''' + 2y'' = 3e^t + 5te^{-t} + e^{-t}\sin t$$

C.E.: $r^4 + 2r^3 + 2r^2 = 0.$ $r^2(r^2 + 2r + 2) = 0$

$r_1 = r_2 = 0.$ $r_{3,4} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t(D\sin t + E\cos t)e^{-t}.$$

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ii) Find the general solution of differential equation $y'' + \omega_0^2 y = \cos\omega_0 t$ where ω_0 is a given constant.

C.E. $r^2 + w_0^2 = 0$ $r_1 = w_0 i.$ $r_2 = -w_0 i.$

$Y_c = c_1 \cos w_0 t + c_2 \sin w_0 t.$

$Y = t(A\cos w_0 t + B\sin w_0 t).$

$Y' = A\cos w_0 t + B\sin w_0 t + t(-Aw_0 \sin w_0 t + Bw_0 \cos w_0 t)$

$Y'' = -Aw_0 \sin w_0 t + Bw_0 \cos w_0 t + (-Aw_0 \sin w_0 t + Bw_0 \cos w_0 t)$

$+ t(-Aw_0^2 \cos w_0 t - Bw_0^2 \sin w_0 t)$

[50]

Sub:

$-2Aw_0 \sin w_0 t + 2Bw_0 \cos w_0 t = \cos w_0 t$

$\Rightarrow A = 0.$ $B = \frac{1}{2w_0}.$

$Y = \frac{t}{2w_0} \sin w_0 t.$

$y = c_1 \cos w_0 t + c_2 \sin w_0 t + \frac{t}{2w_0} \sin w_0 t.$