Last Name: **First name:** **Student ID:**

1. Solve the given differential equations, finding y explicitly if possible.

[7] (a) $\frac{dy}{dt} + 2y = t + e^{-2t}$

$$u = e^{\int p(t)dt} = e^{2t}$$

$$y = \frac{\int u g \, dt}{u} = \frac{\int e^{2t}(t+e^{-2t})dt}{e^{2t}} = \frac{\int (te^{2t}+1)dt}{e^{2t}} = \frac{\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + t + C}{e^{2t}}$$

$$y = \frac{1}{2}t - \frac{1}{4} + te^{-2t} + ce^{-2t}$$

[7] (b) $\frac{dy}{dx} = e^{y}\cos x \,/(y+1)$

$$\int (y+1)e^{-y} dy = \int \cos x \, dx$$

$$-ye^{-y} - 2e^{-y} = \sin x + C$$

[7] (c) $y' = \frac{3y}{3x-y}$

$$v = \frac{y}{x}. \qquad v + xv' = \frac{3v}{3-v}$$

$$xv' = \frac{3v}{3-v} - v = \frac{3v - 3v + v^2}{3-v} = \frac{v^2}{3-v}$$

$$\frac{3-v}{v^2} dv = \frac{1}{x} dx$$

$$\int \frac{3-v}{v^2} = \int \frac{1}{x} dx \qquad -\frac{3}{v} - \ln|v| = \ln|x| + C$$

$$-\frac{3x}{y} - \ln\left|\frac{y}{x}\right| = \ln|x| + C$$

[8] 2. Find the value of y_0 for which the solution of the initial value problem

$$y' - 2y = 4 + e^{-2t}, \quad y(0) = y_0$$

remains finite as $t \longrightarrow +\infty$

$\mu = e^{\int -2dt} = e^{-2t}$. $y(t) = \dfrac{\int \mu g(t)}{\mu} = \dfrac{\int e^{-2t}(4+e^{-2t})}{e^{-2t}}$

$= \dfrac{\int 4e^{-2t} + e^{-4t}}{e^{-2t}} = \dfrac{-2e^{-2t} - \frac{1}{4}e^{-4t} + C}{e^{-2t}} = -2 - \frac{1}{4}e^{-2t} + Ce^{2t}$

$y(0) = y_0 = -2 - \frac{1}{4} + C \quad C = \frac{9}{4} + y_0$

$y(t) = -2 - \frac{1}{4}e^{-2t} + (\frac{9}{4} + y_0)e^{2t} \qquad y_0 = -\frac{9}{4}$

$y(t) = -2 - \frac{1}{4}e^{-2t}$ it is final.

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[8] 3. Find the value of ω for which the given equation is exact and solve the equation using that value of ω

$$(xy^2 + \omega x^2 y)\,dx + (x+y)x^2\,dy = 0$$

$M = xy^2 + \omega x^2 y$. $N = x^3 + x^2 y$.

$M_y = 2xy + \omega x^2$, $N_x = 3x^2 + 2xy$

if $\omega = 3$. $M_y = N_x$.

$F = \int M\,dx + h(y) = \int (xy^2 + 3x^2 y)\,dx + h(y)$

$= \frac{x^2y^2}{2} + x^3 y + h(y)$.

$F_y = N$: $x^2 y + x^3 + h'(y) = x^3 + x^2 y$

$h'(y) = 0$. $h(y) = C$

$F = \frac{x^2y^2}{2} + x^3 y = C$ is a solution

[5] 4. Find an integrating factor to transform the following equation into an exact one.(do **NOT solve the equation**)

$$[4\frac{x^3}{y^2} + \frac{3}{y}]dx + [3\frac{x}{y^2} + 4y]dy = 0.$$

$M = \frac{4x^3}{y^2} + \frac{3}{y}$, $N = \frac{3x}{y^2} + 4y$, $M_y = -8x^3y^{-3} - 3y^{-2}$

$N_x = \frac{3}{y^2}$

$\frac{M_y - N_x}{M} = \frac{-8x^3y^{-3} - 3y^{-2} - \frac{3}{y^2}}{4x^3y^{-2} + 3y^{-1}} = -2y^{-1}$

$\mu' = 2y^{-1}\mu = \frac{2}{y}\mu$. $\mu = e^{2\ln y} = y^2$.

$(4x^3 + 3y)dx + (3x + 4y^3)dy = 0$

it is ~~exact~~, exact.

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[5] 5. Solve the equation

$$y' = r(1 - \frac{y}{K})y,\ y(0) = y_0$$

via the Bernoulli technique.

$y' = ry - \frac{r}{K}y^2$, $y(0) = y_0$

$\Leftrightarrow$ $y^{-2}y' - r\cdot y^{-1} = -\frac{r}{K}$.

Let $V = y^{-1}$, then $V' = -y^{-2}y'$,

Thus $+V' + rV = \frac{r}{K}$, $\mu(t) = e^{rt}$

$V = \frac{\int \mu g\,dt}{\mu} = \frac{\int e^{rt}\cdot(\frac{r}{K})dt}{e^{rt}} = \frac{\frac{1}{K}e^{rt} + C}{e^{rt}}$

$\Rightarrow \frac{1}{y} = \frac{1 + cKe^{-rt}}{K}$, $y(0) = y_0 \Rightarrow C = \frac{1}{y_0} - \frac{1}{K}$, Thus, $y = \frac{K}{1 + (\frac{K}{y_0} - 1)e^{-rt}}$

[3] 6. Solve the equation

$$y'' + \frac{3}{t}y' = 0$$

$v = y'$. $v' + \frac{3}{t}v = 0$ $\frac{dv}{dt} = -\frac{3}{t}v$

$\int \frac{dv}{v} = \int -\frac{3}{t}dt$

$\ln v = -3\ln t + C_1$

$v = C_1 e^{-3\ln t} = C_1 t^{-3}$.

$y = \int v\,dt = \int C_1 t^{-3}dt = \frac{C_1}{-2}t^{-2} + C_2$.

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