Mathematics 2260: ODE (I)

Solution to Assignment 5

1. 1) $\int_0^\infty e^{-st}tdt = \lim_{A\to\infty}\int_0^A e^{-st}tdt$

$= \lim_{A\to\infty}\frac{1}{-s}\int_0^A tde^{-st} = \lim_{A\to\infty}\frac{1}{-s}te^{-st}|_0^A + \frac{1}{s}\int_0^A e^{-st}dt = \frac{1}{s}\int_0^A e^{-st}dt = \frac{1}{s^2},\ s>0.$

2) $\int_0^\infty e^{-st}te^{-at}dt = \int_0^\infty e^{-(s+a)t}tdt = \lim_{A\to\infty}\int_0^A e^{-(s+a)t}tdt = \lim_{A\to\infty}\frac{1}{-(s+a)}\int_0^A tde^{-(s-a)t}$
$= \lim_{A\to\infty}\frac{1}{-(s-a)}te^{-(s+a)t}|_0^A + \frac{1}{s+a}\int_0^A e^{-(s-a)t}dt = \frac{1}{(s+a)^2},\ s>-a.$
3) $\frac{1}{s} - \frac{1}{s}e^{-\pi/2\ s}$

2 1) Using partial fractions we have $\frac{4}{s^2+4s-5} = \frac{4}{6}(\frac{1}{s-1} - \frac{1}{s+5})$. So $L^{-1}(Y(s)) = \frac{4}{6}(e^t - e^{-5t})$.

2)Note that $s^2+2s+5 = (s+1)^2+4$. Thus $\frac{4s+4}{s^2+2s+5} = \frac{4(s+1)}{(s+1)^2+2^2}$. So $L^{-1}(\frac{4s+4}{s^2+2s+5}) = 4e^{-t}\cos 2t$.

3) Using the formula $L(e^{at}f(t)) = F(s-a)$,we have $L^{-1}(5\times 4\frac{3!}{(s-2)^4}) = 20t^3e^{2t}$.

4) First consider $G(s) = \frac{2(s-1)}{s^2-2s+2} = \frac{2(s-1)}{(s-1)^2+1}$. $L^{-1}(G(s)) = 2e^t\cos t$. So $L^{-1}(e^{2s}G(s)) = 2u_2(t)e^{(t-2)}\cos(t-2)$.

3. 1) Taking the Laplace transform of the differential equation, we have

$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 5Y(s) = 0$, or

$Y(s) = \frac{2s+3}{s^2+2s+3} = \frac{2(s+1)+1}{(s+1)^2+4} = \frac{2(s+1)}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}$. So we have

$y(t) = L^{-1}(Y(s)) = 2e^{-t}\cos 2t + \frac{1}{2}e^{-t}\sin 2t$.

2). $s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - 4(s^3Y(s) - s^2y(0) - sy'(0) - y''(0))$
$+6(s^2Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) + Y(s) = 0$
$\Longrightarrow (s^4 - 4s^3 + 6s^2 - 4s + 1)Y(s) = s^2 - 4s + 7$.

$$\begin{aligned} Y(s) &= \frac{s^2-4s+7}{(s-1)^4} = \frac{(s-1)^2 - 2(s-1) + 4}{(s-1)^4} \\ &= \frac{1}{(s-2)^2} - \frac{2}{(s-1)^3} + \frac{4}{(s-1)^4}. \end{aligned}$$

So we have $y(t) = L^{-1}(Y(s)) = te^t - t^2e^t + \frac{2}{3}t^3e^t$.

3). $f(t) = 1 - u_{\pi/2}(t)$. $L(f(t)) = \frac{1}{s} - \frac{1}{s}e^{-\pi/2\ s}$.
$\Rightarrow s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{1}{s} - \frac{1}{s}e^{-\pi/2\ s}$
$\Longrightarrow (s^2+1)Y(s) = 1 + \frac{1}{s} - \frac{1}{s}e^{-\pi/2\ s}$ and
$Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{1}{s(s^2+1)}e^{-\pi/2\ s} = \frac{1}{s^2+1} + \left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-\pi/2\ s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$
$y(t) = L^{-1}(Y(s)) = \sin t + 1 - \cos t - u_{\pi/2}(t) + u_{\pi/2}(t)\cos(t - \pi/2)$.
4). $s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s}e^{-2s}$,that is
$(s^2+3s+2)Y(s) = 1 + \frac{1}{s}e^{-2s}$.
$Y(s) = \frac{1}{(s+1)(s+2)} + \frac{1}{s(s+1)(s+2)}e^{-2s} = \frac{1}{(s+1)} - \frac{1}{s-2} + \left(\frac{1/2}{s} + \frac{-1}{s+1} + \frac{1/2}{s+2}\right)e^{-2s}.$
$y(t) = L^{-1}(Y) = e^{-t} - e^{-2t} + u_2(t)\left(\frac{1}{2} - e^{-(t-2)} + \frac{1}{2}e^{-2(t-2)}\right).$

4. 1) $x_1 = u,\ x_2 = u'$. It follows $x_1' = x_2$ and $x_2' = u'' = -2u - \frac{1}{2}u' = -2x_1 - \frac{1}{2}x_2$. The system is

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - \frac{1}{2}x_2 \end{cases}.$$

2)Similarly to 1), we have

$$\begin{cases} x_1' = x_2 \\ x_2' = -2x_1 - \frac{1}{2}x_2 + 4\cos t \end{cases}.$$

3). Rewrite it as $u'' = -\frac{1}{t}u' - (1 - \frac{1}{t^2})u$. Set $x_1 = u$ and $x_2 = u'$ to have $x_1' = x_2$ and $x_2' = u'' = -\frac{1}{t}u' - (1 - \frac{1}{t^2})u = -\frac{1}{t}x_2 - (1 - \frac{1}{t^2})x_1$. The system is

$$\begin{cases} x_1' = x_2 \\ x_2' = -(1 - \frac{1}{t^2})x_1 - \frac{1}{t}x_2 \end{cases}.$$

4). Set $x_1 = u,\ x_2 = u',\ x_3 = u''$. Thus we have

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_1 \end{cases}.$$

5 1)From the first equation we have $x_2 = x_1' + 2x_1$. Substitution into the second equation we have $(x_1' + 2x_1)' = x_1 - 2(x_1' + 2x_1)$, that is,

$$x_1'' + 4x_1' + 3x_1 = 0.$$

The C.E. is $r^2 + 4r + 3 = 0,\ r_1 = -1,\ r_2 = -3$. $x_1 = c_1e^{-t} + c_2e^{-3t}$. Using $x_2 = x_1' + 2x_1$ gives $x_2 = (c_1e^{-t} + c_2e^{-3t})' + 2\,(c_1e^{-t} + c_2e^{-3t}) = c_1e^{-t} - c_2e^{-3t}$. $c_1 = 1, c_2 = 0$.

2). $x_2 = x_1'/2 + x_1/4$. upon substitution we have $x_1'' + x_1'/4 = -2x_1 - (x_1'/2 + x_1/4)/2$, that is, $x_1'' + x_1' + \frac{17}{4}x_1 = 0$. The general solution for $x_1(t)$ is $x_1(t) = e^{-t/2}(c_1\cos 2t + c_2\sin 2t)$. By using $x_2 = x_1'/2 + x_1/4$, we obtain $x_2(t) = e^{-t/2}(-c_1\sin 2t + c_2\cos 2t)$. $c_1 = 2, c_2 = 2$.

6. 1)Take the Laplace transform:$X_1 = L(x_1), X_2(s) = L(x_2)$.Then

$$\begin{cases} sX_1 - 1 = -2X_1(s) + X_2(s) \\ sX_2(s) - 1 = X_1(s) - 2X_2(s) \end{cases}$$

Solve the above system to have

$$X_1(s) = \frac{1}{s+1},\ X_2(s) = \frac{1}{s+1}$$

So

$$x_1(t) = e^{-t}, \quad x_2(t) = e^{-t}.$$

2)Take the Laplace transform:$X_1 = L(x_1), X_2(s) = L(x_2)$. Then we have

$$\begin{cases} sX_1(s) - 2 = \frac{-1}{2}X_1 + 2X_2 \\ sX_2(s) - 2 = -2X_1 - \frac{1}{2}X_2 \end{cases}$$

or

$$\begin{cases} (s + \frac{1}{2})X_1 - 2X_2 = 2 \\ (s + \frac{1}{2})X_2 + 2X_1 = 2 \end{cases}.$$

Solve the above system to have

$$X_1 = \frac{2(s+\frac{1}{2})+4}{(s+\frac{1}{2})^2+4}, \ X_2 = \frac{2(s+\frac{1}{2})-4}{(s+\frac{1}{2})^2+4}$$

So take the inverse laplace transform to have

$$x_1(t) = e^{-t/2}(2\cos 2t + 2\sin 2t), \ \ x_2(t) = e^{-t/2}(-2\sin 2t + 2\cos 2t).$$